Generic Existence of Local Political Equilibrium

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Abstract. The paper presents a model of multi-party, "spatial" competition under proportional rule with both electoral and coalitional risk. Each party consists of a set of delegates with heterogeneous policy preferences. These delegates choose one delegate as leader or agent. This agent announces the policy declaration (or manifesto) to the electorate prior to the election. The choice of the agent by each party elite is assumed to be a local Nash equilibrium to a game form $\tilde{g}$. This game form encapsulates beliefs of the party elite about the nature of both *electoral risk* and the post-election *coalition bargaining game*. It is demonstrated, under the assumption that $\tilde{g}$ is smooth, that, for almost all parameter values, a locally isolated, local Nash equilibrium exists.

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1 Introduction

Attempts to model party behavior in representative democracies can be classified into two categories. The usual result in the first, "Downsian" (Downs [14]) approach is that parties converge to the center of an electoral distribution (See for example, Calvert [8]). The second approach emphasizes post-election party negotiation, taking the political strengths of parties as given. This "Rikerian" framework (Riker [25]) focuses on post-election bargaining. Both approaches are incomplete, since parties must pay attention to both electoral and post-election coalition risks.

The purpose here is to present the optimization problem for party elite in order to study how parties balance electoral and policy objectives. The key assumption is that each party is a coalition of heterogeneous elite actors. Each actor has "Euclidean" policy preferences defined on a policy space, Z, typically of at least two dimensions. The set of parties is labeled P, and $i \in P$ denotes an arbitrary party in this set. Each party i "strategically" chooses a policy position, z_i, as a *best response* declaration to the other parties' declarations, based on its beliefs about the nature of the political game. To solve the problem of *credible commitment* (Banks [3]) implicit in any model of strategic

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choice where preferences are defined in terms of policies, I suppose that each party, i, chooses, as a leader, that delegate whose ideal policy position coincides with the party's best strategic position, z_i. Such a decision signals to the electorate that the party leader will attempt to implement that position in coalition negotiation.

Parties face *electoral risk* in the pre-election game. A smooth stochastic operator Ψ maps the vector, $z = (z_1, \dots, z_p)$, of strategic declarations into a collection of probabilities $\{\pi_t(z)\}$, where t is finite and indexes the various post-election states. Parties also face post-election, *coalitional risk.* For each realization t of the election, there is a family of decisive coalitions, denoted by $\mathcal{D}_t$. All possible outcomes in state t, both in policy and distribution of government perquisites, are represented by a "smooth" lottery $\tilde{g}_t(z)$. I assume that $\tilde{g}_t(z)$ is generated as a selection from an object called the *heart*, $\mathcal{H}_t(z)$. The *heart* is a "solution" notion (Schofield [29]) derived from the *uncovered set* (Miller [22], McKelvey [20], Cox [11], Banks, Duggan and Le Breton [6,7]). When combined with the stochastic election operator, the combined lottery $\tilde{g}(z) = \{\pi_t(z), \tilde{g}_t(z)\}$ generates a *game form* $\tilde{g}$ which is smoothly dependent on z.

For this model, I give intuitive, and (I believe) plausible arguments that the "centripetal" tendency induced by Downsian vote maximizing is countered by a "centrifugal" tendency generated by coalition risk (Cox [12]). The framework proposed is an attempt to generalize a two party model proposed by Cox [10] and a three party model proposed by Austen-Smith and Banks [1]). An attractive feature of the multiparty model presented here is that it provides a theoretical explanation for the observation that parties do not converge to the electoral center [34,35].

2 The Structure of the Political Game

I analyze the following sequential game:

(i) Each party, i, is characterized by the preferences of its *principal*, also labeled i, who has Euclidean policy preferences, q_i, on the policy space Z derived from the smooth spatial utility $u_i(y) = - \| y - x_i \|^2$, with "bliss" point x_i. The *principal* also has linear preferences over perquisites. So, if i belongs to a government coalition that chooses policy $y \in Z$, and party i receives perquisite δ_i, then the overall utility to i is $u_i(y, \delta_i) = - \| y - x_i \|^2 + \alpha_i \delta_i$.

(ii) Each *principal* chooses a leader to act as an agent for the party. The leader is described by an ideal point $z_i \in Z$. Party i declares z_i as the intended policy for the party, prior to the election. Let Z^p be the set of all possible vectors of declarations of the p parties. It is common knowledge to all delegates that the electoral response $\Psi(z)$ is a stochastic function of the vector of declarations, $z \in Z^p$. A realization of the electoral response is a vector $e = (e_1, \dots, e_p)$ of seat shares.

I assume that any vector, e, of seat shares determines a particular family, say $\mathcal{D}_t$, of decisive coalitions under the specific rules of the Parliament. Such a family is called a coalition structure. In general a coalition M belongs to $\mathcal{D}_t$ only if $\sum_{i \in M} e_i > \frac{1}{2}$. (In principle, I allow the possibility that a coalition M controls half the seats but is not decisive.) I use $\mathcal{D}_1, \ldots, \mathcal{D}_T$ to denote all possible coalition structures. At the vector z, the pre-election probability, under Ψ, that coalition structure $\mathcal{D}_t$ occurs is $\pi_t(z)$.

Thus beliefs about electoral response can be represented by an electoral probability function $\pi : Z^p \to \Delta$, where $\pi = (\pi_1, \ldots, \pi_T)$ and Δ is the $(T-1)$-dimensional simplex.

(iii) For each $\mathcal{D}_t$, and given z, the set of possible government policy outcomes and the set of distributions of coalition perquisites is believed by all delegates to lie in a set, called the *generalized heart* $\mathcal{H}_t(z)$. This object is a subset of $W = Z \times \Delta_P$. (See the Technical Section below for a formal definition of the *heart*. It suffices for the moment to note that $\mathcal{H}_t(z)$ is determined by the structure $\mathcal{D}_t$ and by the vector z of party leader positions.) As before, Z is the policy space while Δ_P is the set of all distributions of government perquisites. Coalition bargaining determines a lottery of outcomes $\tilde{g}_t(z)$. The lottery $\tilde{g}_t(z)$ is a probability measure with support $\mathcal{H}_t(z)$. I emphasize that $\tilde{g}_t(z)$ represents common beliefs of the elite over outcomes in state t, at vector z.

(iv) The payoff to party i is given by a payoff function, U_i^g, induced from the *game form* $\tilde{g}$. As above, $\tilde{g}$ induces, at the vector $z \in Z^p$, a lottery $\{\pi_t(z), \tilde{g}_t(z)\}$. The payoff function $U_i^g : Z^p \to \mathbb{R}$ is given by $U_i^g(z) = U_i(\tilde{g}(z)) = \sum_t \pi_t(z)\, U_i^t(z)$, where $U_i^t(z) = U_i(\tilde{g}_t(z))$ is obtained by taking the expectation induced by u_i across the coalition lottery, $\tilde{g}_t(z)$.

Formally, for each z, $\tilde{g}_t(z)$ is a specific Borel probability measure whose support is contained in the heart $\mathcal{H}_t(z)$. I assume that for each principal i, the utility u_i can be re-expressed as a function $u_i : Z \times \Delta_P \to \mathbb{R}$ and this can be extended to the function $U_i : \tilde{W} \to \mathbb{R}$, measurable with respect to the Borel σ-algebra on W. (Throughout this paper, if X is a (subset of a) topological space, then $\tilde{X}$ is the space of probability measures on X endowed with the weak topology. See Parathasathy [24], for details.) Below I assume that $\tilde{g}_t : Z^p \to \tilde{W}$ is differentiable so that for all i the induced function $U_i^t : Z^p \to \mathbb{R}$ is also differentiable. Note that if $g \in \tilde{W}$, then it is a measure on the Borel σ-algebra of W. Thus, for each $U : \tilde{W} \to \mathbb{R}$, I assume that $\int U dg$ is well defined and can be identified with $U(g) \in \mathbb{R}$. In the weak topology a set g_β of measures converges to g if and only if $\int U dg_\beta$ converges to $\int U dg$ for *every* bounded, continuous utility function U on W. For technical reasons I assume that Z and thus W and $\tilde{W}$ are compact sets, under the relevant topologies.

A *Mixed Strategy Nash Equilibrium* (*MNE*) to the game form $\tilde{g}$, at the profile $\{U_i : i \in P\}$, is a vector $\tilde{z}^* = (\tilde{z}_1^*, \ldots, \tilde{z}_p^*) \in \tilde{Z}^p$ (where $\tilde{Z}^p$ is the Borel space of probability measures over Z^p), such that $\forall\ \tilde{z}_i^* \in \tilde{Z}^p$ and $\forall\ i \in P$

$\tilde{U}_i^g(\tilde{z}_1^*,\ldots,\tilde{z}_i,\ldots,\tilde{z}_p^*) > \tilde{U}_i^g(\tilde{z}_1^*,\ldots,\tilde{z}_i^*,\ldots,\tilde{z}_p^*)$ for no $\tilde{z}_i \in \tilde{Z}$. Here $\tilde{U}_i^g$ is the extension of U_i^g to the domain $\tilde{Z}^p$.

A *Local, Pure Strategy Nash Equilibrium* (*LNE*) is a vector $z^* = (z_1^*,\ldots,z_p^*) \in Z^p$ such that $\forall\ i \in P$ there is a neighborhood V_i of z_i^* in Z such that $U_i^g(z_1^*,\ldots,z_i,\ldots,z_p^*) > U_i^g(z_1^*,\ldots,z_i^*,\ldots,z_p^*)$ for no $z_i \in V_i$.

A *LNE* is a *Global Nash Equilibrium*, (*GNE*) if, for each i, the neighborhood, V_i, is, in fact, Z.

It is usual to focus on *MNE* since they are known to exist under typical continuity and compactness assumptions. However, this concept requires agents to randomize across their policy choices. In the model each party principal chooses a leader whose position is a component of a *LNE*. Focusing on *Local Nash Equilibria,* I effectively only allow "small" perturbations of leadership position. Thus, a vector of party positions z^* is an *LNE* if $\forall\ i \in P$ there exist no z_i *close* to z_i^*, such that i may unilaterally switch from z_i^* to z_i and increase U_i^g.

I show that *LNE* "generically" exist, under the assumptions of smoothness. Moreover, the *LNE* generically consist of isolated policy choices.

3 Modeling the Election

I assume an exogenously given set of parties, $P = \{1,\ldots,i,\ldots,p\}$ and set of voters $N = \{1,\ldots,v,\ldots,n\}$. All actors and voters have preferences on the policy space Z. Each voter, v, has a quasi-concave utility function $u_v : Z \rightarrow \mathbb{R}$, which, with little loss of generality, I assume is Euclidean and of the form $u_v(y) = -a_v \parallel y - x_v \parallel^2$. Here x_v is the voter's ideal point and a_v is a positive constant. Each party, i, makes a declaration z_i, so $z = (z_1,\ldots,z_p) \in Z^p$ is the declaration profile. Let Δ_P be the $(p-1)$ dimensional unit simplex. A vector $v = (v_1,\ldots,v_p) \in \Delta_P$ represents the vote shares and a vector $e(v) = (e_1,\ldots,e_p) \in \Delta_P$ represents the seat shares of parties. Together, v and $e(v)$ represent the post-election realizations of the decisions of the electorate. I assume that a response by voter v is defined by a smooth probability function: $\chi_v : Z^p \longrightarrow \Delta_P$. Thus $\chi_v(z) = (\ldots,\psi_{vi}(z),\ldots)$, where $\psi_{vi}(z)$ is the probability that voter v picks party i, at a declared profile z. Related empirical analyses used a multionomial probit model (MNP) to estimate the random variable $\Psi(z)$, whose components are the random variables characterizing the vote shares of the various parties. In this model, the probability $\psi_{vi}(z)$ that v picks i, is given by the probability that the realized utility $u_v(z_i) = -a_v \parallel z_i - x_v \parallel^2 + \epsilon_i$ of v at z_i exceeds $u_v(z_k)\ \forall\ k \neq i$. The MNP model does not assume that the error vector $\epsilon = (\epsilon_1 \ldots \epsilon_p)$ is characterized by a diagonal covariance matrix. Instead, it uses a Bayesian iteration procedure (Chib and Greenberg [9]) to estimate the multinomial covariance matrix. (See Schofield, Martin, *et al.* [34].)

In general $\Psi(z)$ will lie in the space $\tilde{\Delta}_P$ of Borel probability measures on Δ_P. With respect to the weak topology on $\tilde{\Delta}_P$, I suppose that Ψ is smooth.

I interpret Ψ as a proxy for the *beliefs* of party *principals* over the electoral response. I do not demand that each voter behaves stochastically, and though it is plausible that voters' choices are based on strategic reasoning, I do not attempt to model such choice theoretically. Instead I suggest that the elite political agents will form beliefs on such aggregate choices through empirical analysis (such as opinion polls). It is sensible, however, to suppose, for any set of voters whose ideal points lie in a neighborhood V of x_i, that the proportions, which vote for each party, under the vector z of declarations, are described by smooth functions $\{\psi_{v_i}(z)\}$ of the point x_i.

Given Ψ there is an expectation operator $E(\Psi) : Z^p \to \Delta_P$, where $E(\Psi(z)) = E(\Psi)(z)$ is the expectation of the vote shares at profile z. If the voter choice functions were independent, then the expected value of the vote share of the i^{th} party would be the average of $\{\psi_{vi}\}_v$. I denote by $e(\Psi(z))$ the random variable describing the seat shares of parties at z. $E(e(\Psi(z)))$ is the expectation of this vector. In a pure proportional electoral system, the random variables $e(\Psi(z))$ and $\Psi(z)$ will be identical for all z.

Assumption 1: The electoral probability function $\pi : Z^p \to \Delta$ is a common knowledge, smooth, function from Z^p to the simplex Δ (of dimension *T-1*).

In the MNP model, the covariance structure on the errors implies significant variance in the vote share variables. Thus estimating Ψ, and assuming $e(\Psi) \simeq \Psi$, allows an estimation of π. Moreover, the smoothness of the MNP estimator Ψ implies that π is also smooth. Thus the MNP empirical analyses published previously are compatible with the theoretical Assumption 1, that elections are inherently risky, but smoothly determined by leader positions. Note also the implicit assumption that voter beliefs are also smooth in all relevant parameters.

4 Existence of Nash Equilibrium

I model coalition bargaining as a "committee game" among the p party leaders that takes place after the vector z of party declarations and the vector e of election results are known. Consequently, the set $\mathcal{D}_t$ of decisive coalitions is known (a review of formal models of committee bargaining can be found in Austen-Smith and Banks [2]; Laver [18] provides an overview of more empirical analyses of coalition behavior).

Recall that a coalition M of *leaders* belongs to $\mathcal{D}_t$ only if their parties control over half the seats. I assume that each leader (of party i) is described by an ideal point z_i and a linear utility $\alpha_i\delta_i$ on government perquisites. This induces for each i a preference correspondence $q_i(z_i) : Z \times \Delta_P \to Z \times \Delta_P$. Again, Δ_P is the set of possible distributions of government perquisites among party leaders. I assume $\alpha = (\alpha_1, \dots, \alpha_p)$ is common knowledge. Let $q^\alpha(z)$ denote the preference profile generated by the preferences of party leaders. Since the election results are known, I can compute the *generalized* *heart*, $\mathcal{H}_{\mathcal{D}_t}(q^\alpha(z))$ at the coalition structure $\mathcal{D}_t$ and preference profile $q^\alpha(z)$. (Formal properties of this object, regarded as a correspondence, are given in the Technical Section below.) Since $\mathcal{D}_t$ is fixed and $q^\alpha(z)$ is specified, I write this object, the heart, as $\mathcal{H}_t^\alpha(z)$, noting again that it is a subset of $Z \times \Delta_P$. The *heart* is known to be non empty for all $q^\alpha(z)$. As a correspondence from Z^p to $Z \times \Delta_P$, $\mathcal{H}_t^\alpha$ is "lower hemi-continuous" and admits a smooth selection $\tilde{g}_t^\alpha : Z^p \to Z \times \Delta_P$. I assume that the set of coalition bargaining outcomes, in the state $\mathcal{D}_t$ and at the profile z, is given by a lottery from the *generalized heart.* Let $\tilde{\mathcal{H}}_t^\alpha(z)$ be the set of all mixtures (or lotteries) over $\mathcal{H}_t^\alpha(z)$.

Assumption 2. For fixed $\mathcal{D}_t, z$ and α, the outcome of bargaining at the profile $q^\alpha(z)$ is a "common knowledge" lottery $\tilde{g}_t^\alpha(z) \in \tilde{\mathcal{H}}_t^\alpha(z)$. Moreover, as a function $\tilde{g}_t^\alpha : Z^p \to \tilde{Z} \times \tilde{\Delta}_P$ is smooth. Here $\tilde{Z} \times \tilde{\Delta}_P$ is the space of all lotteries over $Z \times \Delta_P$ (again endowed with the weak topology). Here I use $\tilde{g}_t^\alpha$ to remind the reader of the dependence of this "selection" on the parameter α.

One implication of this assumption is that given $\mathcal{D}_t$, and the profile $q^\alpha(z)$, there may exist a point (x, δ) in the *voting core.* I will denote the *voting core* by $E_t^\alpha(z)$. An outcome, (x, δ), where x is a policy point, and δ a distribution of perquisites, is in the *voting core* $E_t^\alpha(z)$ iff (x, δ) is unbeaten: no (y, δ') exists which is preferred by all leaders of a decisive coalition (in $\mathcal{D}_t$). I observe in the Technical Section below that if $E_t^\alpha(z) \neq \phi$ then $\mathcal{H}_t^\alpha(z) = E_t^\alpha(z)$. In fact $\mathcal{H}_t^\alpha(z')$ "converges" to $E_t^\alpha(z)$ as z' converges to z. Thus, if $\tilde{g}_t^\alpha$ is a selection of $\tilde{\mathcal{H}}_t^\alpha$, it must be the case that all components of $\tilde{g}_t^\alpha(z)$ belong to $E_t^\alpha(z)$, whenever $E_t^\alpha(z) \neq \phi$.

I also introduce the *policy heart* $\mathcal{H}_{\mathcal{D}_t}(q(z)) = \mathcal{H}_t^0(z) \subset Z$, obtained by setting all values of α_i equal to zero. When values of the parameters $\{\alpha_i\}$ are sufficiently high, policy compromises have little relevance for bargaining. In this case, party bargaining takes place in the context of a constant sum voting game where the voting core will be typically empty. However, if α is zero, the policy core $E_t^0(z)$ may be non empty. Typically a point in the policy core (if non-empty) will be at the policy position of the *strongest* party in the legislature (see Schofield [30] for the formal definition of "strongest"). Sened [36]) shows that certain constraints on the value of perquisites guarantee that a center *core* party will always be a member of the government coalition.

The function $\tilde{g}_t^\alpha : Z^p \to \tilde{Z} \times \tilde{\Delta}_P$ is the *game form*, conditional on the belief that $\mathcal{D}_t$ is the election outcome. So, I am able to compute, for each party, the expected utility associated with a vector of declarations $z \in Z^p$.

Assumption 3. The expected utility for i at a strategy vector $z \in Z^p$, conditional on $\mathcal{D}_t$, is given by a smooth function $U_i^t(z) : Z^p \to \mathbb{R}$ defined by $U_i^t(z) = U_i^\alpha(z \mid \mathcal{D}_t) = U_i(\tilde{g}_t^\alpha(z))$.

Here U_i^t is computed across elements of the mixture $\tilde{g}_t^\alpha(z)$ using the Euclidean utility function, based on the ideal policy point x_i of party i's principal together with the linear component derived from perquisites. Without loss of generality I assume all x_i lie in the interior of Z. I require this to forbid boundary solutions to the equilibrium condition.

The generalized heart, $\mathcal{H}_t^\alpha(z)$, is intended as a "constraint" on the beliefs of the elite over the post-election bargaining between party leaders, when their preferences are described by the profile $q^\alpha(z)$, and $\mathcal{D}_t$ is fixed. This bargaining can be viewed as a sequence of proposals and counter proposals in $Z \times \Delta_P$, implemented by a political auctioneer. Social choice theorists (Cox [11], McKelvey [20] have identified the "attractor" of such a process with the uncovered set. Since it makes sense to see political bargaining as a continuous process, I regard the *heart*, or "local" uncovered set, as the attractor. Arguments presented elsewhere (Schofield [27,28]) suggest that party leaders will rationally eliminate outcomes outside the *heart* from their calculations. Delegates, uncertain about the precise nature of the bargaining, could assume that the lottery $\tilde{g}_t^\alpha$ is described by a uniform distribution over the *heart*. The *principal* of party i could calculate U_i^t by integrating U_i over $\mathcal{H}_t^\alpha(z)$. However, I formally require only that the beliefs $\tilde{g}_t^\alpha$ generate a smooth function: $U_i^t : Z^p \longrightarrow \mathbb{R}$, for all $i \in P$.

In the absence of electoral risk the LNE can be computed by setting $dU_i^t = 0$ for all party principals. This is essentially the procedure adopted by scholars who model only the post-election stage of this game. To take electoral risk into account I must compute the pre-election expectation of $U_{i\,t=1,\dots T}^t$. Thus, prior to the election, it is common knowledge that the election outcome, conditional on $z \in Z^p$, is a finite electoral lottery $\tilde{g}^\alpha(z) = \{(\pi_t(z), \mathcal{D}_t)\}$. The expected utility U_i^α to party i can be computed for any profile, z, of declarations, and for any smooth game form $\tilde{g}^\alpha$. The utility function for each party, i, is derived from the game form $\tilde{g}^\alpha$, and the expected utility of the party principal. Thus, $U_i^\alpha(z) = U_i(\tilde{g}^\alpha(z)) = \sum_t \pi_t(z) U_i^t(z)$.

By Assumptions 1 and 2, the game form $\tilde{g}^\alpha$ is smooth. By Assumption 3, for each i, $U_i^\alpha : Z^p \to \mathbb{R}$ is differentiable. This allows the derivation of the central formal result. Theorem 1 asserts, for almost all α and almost any vector of ideal points, $x \in Z^p$, of party principals, that the LNE for the *game form* $\tilde{g}^\alpha$ exists and consists of locally isolated points.

Theorem 1. *Under Assumptions 1, 2, 3, for almost all α and for almost all $x \in Z^p$ the LNE of the game form $\tilde{g}^\alpha$ is non empty and locally isolated.*

Each LNE, z^*, determines a list of leadership choices, and party declarations, all in Z, which is in equilibrium with respect to the pre-election beliefs of the delegates. Given the assumptions on differentiability (and thus continuity) of the utility functions of party principals, a standard result (Glicksberg [15]) implies existence of a MNE. I state this result for completeness:

Theorem 2. *Under Assumptions 1, 2, and 3, then, $\forall$ α, and $\forall$ $x \in Z^p$ the MNE of $\tilde{g}^\alpha$ is non empty.*

The proof of Theorem 1 is presented in the Technical Section below. The proof of Theorem 2 follows by standard results. Schofield and Parks [35] study the nature of the LNE under structural assumptions on the selection function.

5 Technical Section: Formal Definitions and Proof of Theorem 1

A (strict) *preference* Q on a set, or space, W is a correspondence $Q : W \to 2(W)$ where $2(W)$ stands for the family of all subsets of W (including the empty set ϕ). W is a *Fan space* if it is a compact convex subset of a linear topological space of finite dimension.

Definition 1. (i) Let $Q : W \to 2(W)$ be a preference correspondence on the space W. The *core* of Q is $E(Q) = \{x \in W : Q(x) = \phi\}$.

(ii) The covering correspondence, $\tilde{Q}$ of Q is defined by $y \in \tilde{Q}(x)$ iff $y \in Q(x)$ and $Q(y) \subset Q(x)$. Say *y covers x*. The *uncovered set*, $\tilde{E}(Q)$ of Q, is $\tilde{E}(Q) = E(\tilde{Q}) = \{x \in W : \tilde{Q}(x) = \phi\}$.

(iii) If W is a topological space, then $x \in W$ is *locally covered* (under Q) iff for any neighborhood V of x in W, there exists $y \in V$ such that $y \in Q(x)$ and $V \cap Q(y) \subset V \cap Q(x)$. If x is not locally covered, then write $\hat{Q}(x) = \phi$.

The *heart* of Q, written $\mathcal{H}(Q)$, is defined by $\mathcal{H}(Q) = \{x \in W : \hat{Q}(x) = \phi\}$.

A preference Q is convex iff for all x, the preferred set $Q(x)$ of x is strictly convex. The preference is continuous if it is induced from a continuous utility function. For continuous preferences the heart will contain the uncovered set and both will be non empty. Moreover if $E(Q)$ is non-empty, then it is contained in both $\tilde{E}(Q)$ and $\mathcal{H}(Q)$ and if Q is convex, then all three sets are identical. It can be shown that if $E(Q) \neq \phi$ and $Q' \to Q$ in an appropriate topological sense, then $\mathcal{H}(Q') \to E(Q)$ in a set theoretic sense. Formally it is possible to find a sequence $Q' \to Q$ and $z' \in \mathcal{H}(Q')$ such that $z' \to E(Q)$.

Now let $Q^*(W)^P$ stand for all "smooth" convex preference profiles for the society P. Thus $q \in Q^*(W)^P$ means $q = (q_1, \dots, q_p)$ where each q_i is a convex preference, induced from a smooth "quasi-concave" utility function, u_i.

Definition 2. Let $\mathcal{D}$ be a fixed voting rule, and W a Fan space. Let $q \in Q^*(W)^P$ be a smooth preference profile. Define $\sigma_{\mathcal{D}}(q) = \cup_{M \in \mathcal{D}}\{\cap_{i \in M} q_i\} : W \to 2(W)$ to be the preference correspondence induced by $\mathcal{D}$ at q.

The *core* of $\mathcal{D}$ at q, written $E_{\mathcal{D}}(q)$, is $E(\sigma_{\mathcal{D}}(q))$.

The *heart* of $\mathcal{D}$ at q, written $\mathcal{H}_{\mathcal{D}}(q)$, is defined to be $\mathcal{H}(\sigma_{\mathcal{D}}(p))$. The *uncovered set* of $\mathcal{D}$ at p, written $\tilde{E}_{\mathcal{D}}(p)$, is $\tilde{E}(\sigma_{\mathcal{D}}(p))$.

The *Pareto set* of the profile q is $E_P(q) = E(\sigma_P(q))$ where $\sigma_P(q) : \{\cap_{i \in P} q_i\} : W \to 2(W)$ is the Pareto, or strict unanimity, preference correspondence.

A correspondence $Q : W \to Y$ is lower hemi continuous (lhc) with respect to topologies on W, Y iff for any open $V \in Y$ the set $\{x \in W : Q(x) \cap V \neq \phi\}$ is open in W. A continuous selection g for Q is a function $g : W \to Y$, continuous with respect to the topologies on W, Y such that $g(x) \in Q(x)\ \forall\ x \in W$, whenever $Q(x) \neq \phi$. Schofield [31–33] has shown that the heart is non empty, Paretian and a lhc correspondence, with respect to a C^1–topology on $Q^*(W)^P$.

The following three propositions summarize the technical properties of the *heart* correspondence.

Proposition 1. *Let W be a Fan space, and $\mathcal{D}$ any voting rule. Then $\mathcal{H}_{\mathcal{D}} : Q^*(W)^P \to 2(W)$ is lhc. Moreover, for any $q \in Q^*(W)^P, \mathcal{H}_{\mathcal{D}}(q)$ is closed, non empty and is a subset of the Pareto set $E_P(q)$. If $E_{\mathcal{D}}(q) \neq \phi$ then there is a sequence $z' \to E_{\mathcal{D}}(q)$ where $z' \in \mathcal{H}_{\mathcal{D}}(q')$ and $q' \to q$ in the C^1–topology.*

To use the results to model coalition bargaining, I assume that the choice of the *leader* (or *agent*) for party i determines the declaration z_i of the party. As before, I assume the leader has utility function $u_i(y, \delta_i) = - \| y - z_i \|^2 + \alpha_i \delta_i$ which generates a smooth, strictly convex preference correspondence $q_i^{\alpha_i}(z_i) : Z \times \Delta_P \to Z \times \Delta_P$.

The Pareto set in $Z \times \Delta_P$ is the unanimity choice of these preferences. For a fixed voting rule $\mathcal{D}_t$, using Definition 2, I can define the *heart* of the voting rule on the space $W = Z \times \Delta_P$ as $\mathcal{H}_{\mathcal{D}_t}(q^\alpha(z))$. However, since the profile q is fully specified by the vectors α and z, I may write this object as $\mathcal{H}_t^\alpha(z)$. Proposition 1 can then be used to show that $\mathcal{H}_t^\alpha$ is lhc and converges to the voting core $E_t^\alpha(z)$, if it is non empty. Lower hemi-continuity of $\mathcal{H}_t^\alpha$ allows use of Michael's Selection Theorem (Michael [21]) to show existence of a selection, g_t^α. Moreover, since $\mathcal{H}_t^\alpha$ is lower hemi-continuous with respect to a C^1-topology (Schofield [31]) on smooth utility profiles, the selection g_t^α can be chosen to be differentiable.

Proposition 2. *Let Z be a compact convex subset of $\mathbb{R}^w$, endowed with the Euclidean topology, and let Z^p be the product space. Then for any voting rule, $\mathcal{D}_t$, and any parameter α, $\mathcal{H}_t^\alpha : Z^p \to 2(Z \times \Delta_P)$ is lhc.*

Moreover, for each $z \in Z^p, \mathcal{H}_t^\alpha(z)$ is a closed, non empty subset of the Pareto set in $Z \times \Delta_P$. Thus, $\mathcal{H}_t^\alpha$ admits a smooth, Paretian selection g_t^α (where g_t^α is differentiable in z).

Now let $\tilde{Z} \times \tilde{\Delta}_P$ be the set of all lotteries over $\tilde{Z} \times \tilde{\Delta}_P$, endowed with the weak topology. Let $\tilde{\mathcal{H}}_t^\alpha : Z^p \to 2(\tilde{Z} \times \tilde{\Delta}_P)$ be the extension of the heart to this space. Then lhc of $\mathcal{H}_t^\alpha$ implies lhc of $\tilde{\mathcal{H}}_t^\alpha$ (Schofield [32]).

Proposition 3. *For a fixed voting rule, $\mathcal{D}_t$, there exists a smooth selection $\tilde{g}_t^\alpha : Z^p \to \tilde{Z} \times \tilde{\Delta}_P$ of $\tilde{\mathcal{H}}_t^\alpha$.*

Proposition 3 provides a justification for Assumption 2.

I now show that *LNE* exist and are "generically" locally isolated. By "generically", I mean for almost all values of $\{\alpha\}$ and ideal points $x \in Z^p$ of the party principals. (See Schofield [26], for the formal topological definition of "generic".) I use the terminology *Critical Nash Equilibrium* (*CNE*) for a vector $z^* \in Z^p$ which satisfies the condition $\frac{dU_i}{dz_i} = 0$, for all i. Let T_i^α be the subset of Z^p where the first order condition for U_i is satisfied.

Theorem 3. *For each $\tilde{g}_t^\alpha$ and for almost all $\alpha, x \in Z^p$, the LNE of the game form $\tilde{g}^\alpha$ is non empty and locally isolated.*

Proof. For each $z \in Z^p, z_i$ is chosen to satisfy the first order condition $\frac{dU_i}{dz_i} = 0$. By the inverse function theorem T_i^α is generically a smooth manifold of dimension $(p-1)\dim(Z)$. By the Thom Transversality Theorem, the intersection $\bigcap_{i \in P} T_i^\alpha$ is generically of codimension $p\dim(Z)$ in Z^p. (See Golubitsky and Guillemin [16] and Hirsch [17] for the formal statement of the Theorem, and Schofield [26] and Banks [4] for applications.) But Z^p has dimension $p\dim(Z)$. Since the $CNE \equiv \bigcap_{i \in P} T_i^\alpha$, this shows the *CNE* is generically of dimension 0. That is, it consists of locally isolated points. I am now able to construct a gradient field μ on Z^p whose zeros consist precisely of the *CNE* (see Schofield [32], for this construction). Since $Z = W \times \Delta$, it is homeomorphic to the ball, and thus has Euler characteristic 1. Since all elements of the selection are contained in the Pareto set, this field points inward on the boundary of Z^p. The Morse inequalities (Dierker [13]; Milnor [23]) imply that there must be at least one critical point of μ whose index is maximal. This point corresponds to a *LNE*. □

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