1 Technical Appendix

1.1 The Spatial Valence Model

Instead of using a model based on committee bargaining, as in the first section of this chapter we can instead adapt Hinich’s 1977 earlier stochastic model of voting.

Definition 1. The Stochastic Vote Model \( E(\lambda, \mu, \beta; \Psi) \) with Activist Valence. Voters are characterized by a distribution, \( \{ x_i \in W : i \in V \} \), of voter ideal points for the members of the electorate, \( V \), of size \( v \). We assume that \( W \) is an open, convex subset of Euclidean space, \( \mathbb{R}^w \), with \( w \) finite. Each of the parties in the set \( N = \{ 1, \ldots, j, \ldots, n \} \) chooses a policy, \( z_j \in W \), to declare. Let \( z = (z_1, \ldots, z_n) \in W^n \) be a typical vector of party policy positions.

Given \( z \), each voter, \( i \), is described by a vector

\[
\mathbf{u}_i(x_i, \mathbf{z}) = (u_{i1}(x_i, z_1), \ldots, u_{ip}(x_i, z_n)), \quad \text{where}
\]

\[
u_{ij}(x_i, z_j) = \lambda_j + \mu_j(z_j) - \beta||x_i - z_j||^2 + \epsilon_j
\]

\[
u_{ij}(x_i, z_j) = \epsilon_j.
\]

Here \( u_{ij}(x_i, z_j) \) is the observable component of utility. The term, \( \lambda_j \), is the fixed or exogenous valence of party \( j \), while the function \( \mu_j(z_j) \) is the component of valence generated by activist contributions to party \( j \). The term \( \beta \) is a positive constant, called the spatial parameter, giving the importance of policy difference defined in terms of the Euclidean norm, \( || \cdot || \), on \( W \). The error vector \( \epsilon = (\epsilon_1, \ldots, \epsilon_j, \ldots, \epsilon_n) \) is generated by the Type I Extreme Value Distribution, also known as the multinomial logit.

Voter behavior is modeled by a probability vector. The probability that a voter \( i \) chooses party \( j \) at the vector \( \mathbf{z} \) is

\[
\rho_{ij}(\mathbf{z}) = \Pr[[u_{ij}(x_i, z_j) > u_{il}(x_i, z_l)], \text{ for all } l \neq j]. \quad (1)
\]

\[
\rho_{ij}(\mathbf{z}) = \Pr[\epsilon_l - \epsilon_j < u_{ij}^*(x_i, z_j) - u_{il}^*(x_i, z_l)], \text{ for all } l \neq j. \quad (2)
\]

Here \( \Pr \) stands for the probability operator generated by the distribution assumption on \( \epsilon \).

The expected vote-share of agent \( j \) generated by the model \( E(\lambda, \mu; \beta; \Psi) \) is

\[
E_j(\mathbf{z}) = \frac{1}{v} \sum_{i \in V} \rho_{ij}(\mathbf{z}). \quad (3)
\]

Definition 2. The Type I Extreme Value Distribution, \( \Psi \).

The cumulative distribution, \( \Psi \), has the closed form

\[
\Psi(x) = \exp[-\exp[-x]],
\]

with probability density function

\[
\psi(x) = \exp[-x] \exp[-\exp[-x]]
\]

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With this distribution assumption, it follows, for each voter \( i \), and party \( j \), that
\[
\rho_{ij}(\mathbf{z}) = \frac{\exp[u_{ij}^*(x_i, z_j)]}{\sum_{k=1}^{n} \exp[u_{ik}^*(x_i, z_k)]}.
\]

(4)


(i) A strategy vector \( \mathbf{z}^*=(z_1^*, \ldots, z_{j-1}^*, z_j^*, z_{j+1}^*, \ldots, z_n^*) \in W^n \) is a local strict Nash equilibrium (LSNE) for the function \( E : W^n \to \mathbb{R}^n \) iff, for each party \( j \in N \), there exists a neighborhood \( W_j \) of \( z_j^* \) in \( W \) such that
\[
E_j(z_1^*, \ldots, z_{j-1}^*, z_j^*, z_{j+1}^*, \ldots, z_n^*) > E_j(z_1^*, \ldots, z_j, \ldots, z_n^*)
\]
for all \( z_j \in W_j - \{z_j^*\} \).

(ii) A strategy vector \( \mathbf{z}^*=(z_1^*, \ldots, z_{j-1}^*, z_j^*, z_{j+1}^*, \ldots, z_n^*) \) is a local weak Nash equilibrium (LNE) iff, for each agent \( j \), there exists a neighborhood \( W_j \) of \( z_j^* \) in \( W \) such that
\[
E_j(z_1^*, \ldots, z_{j-1}^*, z_j^*, z_{j+1}^*, \ldots, z_n^*) \geq E_j(z_1^*, \ldots, z_j, \ldots, z_n^*)
\]
for all \( z_j \in W_j \).

(iii) A strategy vector \( \mathbf{z}^*=(z_1^*, \ldots, z_{j-1}^*, z_j^*, z_{j+1}^*, \ldots, z_n^*) \) is a strict or weak, pure strategy Nash equilibrium (PSNE or PNE) iff \( W_j \) can be replaced by \( W \) in (i) and (ii) respectively.

Obviously if \( \mathbf{z}^* \) is an LSNE or a PNE it must be an LNE, while if it is a PSNE then it must be an LSNE. We use the notion of LSNE to avoid problems with the degenerate situation when there is a zero eigenvalue to the Hessian. The weaker requirement of LNE allows us to obtain a necessary condition for \( \mathbf{z}^* \) to be an LNE and thus a PNE, without having to invoke concavity. Of particular interest is the joint mean vector
\[
x^* = \frac{1}{v} \sum_{i \in V} x_i.
\]

(5)

We first transform coordinates so that in the new coordinate system, \( x^* = 0 \). We shall refer to \( \mathbf{z}_0 = (0, \ldots, 0) \) as the joint origin.

Theorem 1 below shows that \( \mathbf{z}_0 = (0, \ldots, 0) \) will generally not satisfy the first-order condition for an LSNE, namely that the differential of \( E_j \), with respect to \( z_j \) be zero. However, if the activist valence function is identically zero, so that only exogenous valence is relevant, then the first-order condition will be satisfied. On the other hand, Theorem 2 shows that there are necessary and sufficient conditions for \( \mathbf{z}_0 \) to be an LSNE in a model without activist valence.
A corollary of Theorem 2 gives these conditions in terms of a “convergence coefficient” determined by the Hessian of party 1, with the lowest valence.

It follows from the model that for voter \(i\), with bliss point, \(x_i\), the probability, \(\rho_{ij}(z)\), that \(i\) picks \(j\) at \(z\) is given by

\[
\rho_{ij}(z) = \left[1 + \sum_{k \neq j} \exp(f_{jk})\right]^{-1},
\]

where \(f_{jk} = \lambda_k + \mu_k(z_k) - \lambda_j - \mu_j(z_j) + \beta \|x_i - z_j\|^2 - \beta \|x_i - z_k\|^2\).

Theorem 1 below shows that the first-order condition for \(z_j^*\) to be an LSNE is that it be a balance solution.

**Definition 4: The Balance Solution for the Model \(E(\lambda, \mu, \beta; \Psi)\)**

(i) Let \([\rho_{ij}(z)] = [\rho_{ij}]\) be the matrix of voter probabilities at the vector \(z\), and let

\[
[\alpha_{ij}] = \frac{\rho_{ij} - \rho_{ij}^2}{\sum_k (\rho_{kj} - \rho_{kj}^2)}
\]

be the matrix of coefficients. The balance equation for \(z_j^*\) is given by expression

\[
z_j^* = \frac{1}{2\beta} \frac{d\mu_j}{dz_j} + \sum_{i=1}^v \alpha_{ij} x_i.
\]

(ii) The vector \(\sum_i \alpha_{ij} x_i\) is called the weighted electoral mean for party \(j\), and can be written

\[
\sum_{i=1}^v \alpha_{ij} x_i = \frac{d\xi_j^*}{dz_j}.
\]

(iii) The balance equation for party \(j\) can then be rewritten as

\[
\left[\frac{d\xi_j^*}{dz_j} - z_j^*\right] + \frac{1}{2\beta} \frac{d\mu_j}{dz_j} = 0.
\]

(iv) The bracketed term on the left of this expression is termed the marginal electoral pull of party \(j\) and is a gradient vector pointing towards the weighted electoral mean. This weighted electoral mean is that point where the electoral pull is zero. The vector \(\frac{d\mu_j}{dz_j}\) is called the marginal activist pull for party \(j\).

(v) If the vector \(z^* = (z_1^*, \ldots, z_j^*, \ldots, z_n^*)\) satisfies the set of balance equations, \(j = 1, \ldots, n\), then call \(z^*\) the balance solution.
Theorem 1 is proved in Schofield (2006).

**Theorem 1.** Consider the electoral model $E(\lambda, \mu; \beta; \Psi)$ based on the Type I extreme value distribution, and including both exogenous and activist valences. The first-order condition for $z^*$ to be an LSNE is that it is a balance solution. If all activist valence functions are highly concave, in the sense of having negative eigenvalues of sufficiently great magnitude, then the balance solution will be a PNE.

In the case that all activist valence functions $\{\mu_j\}$ are identically zero, we write the electoral model as $E(\lambda; \beta; \Psi)$. Then by Theorem 1, the coefficients, $\alpha_{ij}$, are independent of $i$. Thus, when there is only exogenous valence, the balance condition gives

$$z_j^* = \frac{1}{v} \sum_{i=1}^{v} x_i. \tag{10}$$

By a change of coordinates we can choose $\Sigma x_i = 0$. In this case, the marginal electoral pull is zero at the origin and the joint origin $z_0 = (0, \ldots, 0)$ satisfies the first-order condition.

To characterize the variation in voter preferences, we represent in a simple form the covariance matrix, $\nabla_0$, given by the distribution of voter ideal points.

**Definition 5. The Electoral Covariance Matrix, $\nabla_0$.**

Let $W = \mathbb{R}^w$ be endowed with a system of coordinate axes $r = 1, \ldots, w$. For each coordinate axis let $\xi_r = (x_{1r}, x_{2r}, \ldots, x_{vr})$ be the vector of the $r^{th}$ coordinates of the set of $v$ voter bliss points. The scalar product of $\xi_r$ and $\xi_s$ is denoted by $(\xi_r, \xi_s)$.

(i) The symmetric $w \times w$ electoral covariance matrix about the origin is denoted $\nabla_0$ and is defined by

$$\nabla_0 = \frac{1}{v} \sum_{r=1}^{w} [(\xi_r, \xi_s)]_{s=1}^{w}. \tag{11}$$

(ii) Let $(\sigma_r, \sigma_s) = \frac{1}{v} (\xi_r, \xi_s)$ be the electoral covariance between the $r^{th}$ and $s^{th}$ axes, and $\sigma_s^2 = \frac{1}{v} (\xi_s, \xi_s)$ be the electoral variance on the $s^{th}$ axis, with

$$\sigma^2 = \sum_{s=1}^{w} \sigma_s^2 = \frac{1}{v} \sum_{s=1}^{w} (\xi_s, \xi_s) = trace(\nabla_0)$$

the total electoral variance.

At the vector $z_0 = (0, \ldots, 0)$ the probability $\rho_{ij}(z_0)$ that $i$ votes for party $j$ is independent of $i$, and is given by

$$\rho_j = \left[ 1 + \sum_{k \neq j} \exp [\lambda_k - \lambda_j] \right]^{-1}. \tag{11}$$
Definition 6: The Convergence Coefficient of the Model $\mathbb{E}(\lambda; \beta; \Psi)$.

The characteristic matrix for party $j$ is

$$C_j = [2A_j \nabla_0 - I],$$

where $I$ is the $w$ by $w$ identity matrix.

(iii) The convergence coefficient of the model is

$$c(\lambda; \beta; \Psi) = 2\beta[1 - 2\rho_1]\sigma^2 = 2A_1\sigma^2.$$  

At the vector $z_0 = (0, \ldots, 0)$ the probability $\rho_{ij}(z_0)$ that $i$ votes for party $j$ is independent of $i$, and is given by (11). Thus, if all valences are identical then $\rho_j = \frac{1}{n}$, for all $j$, as expected. The effect of increasing $\lambda_j$, for $j \neq 1$, is clearly to decrease $\rho_1$, and therefore to increase $A_1$, and thus $c(\lambda; \beta; \Psi)$. Schofield (2007) proves the following theorem.

Theorem 2: The necessary condition for the joint origin to be an LSNE in the model $\mathbb{E}(\lambda; \beta; \Psi)$ is that the characteristic matrix

$$C_1 = [2A_1 \nabla_0 - I]$$

of the party 1, with lowest valence, has negative eigenvalues.  

Theorem 2. immediately gives the following corollaries:

Corollary 3. Consider the model $\mathbb{E}(\lambda; \beta; \Psi)$. In the case that $X$ is $w$-dimensional, then the necessary condition for the joint origin to be an LNE is that $c(\lambda; \beta; \Psi) \leq w$.

Ceteris paribus, an LNE at the joint origin is “less likely” the greater are the parameters $\beta$, $\lambda_p - \lambda_1$ and $\sigma^2$.

Corollary 4. In the two-dimensional case, a sufficient condition for the joint origin to be an LSNE for the model $\mathbb{E}(\lambda; \beta; \Psi)$ is that $c(\lambda; \beta; \Psi) < 1$.

It is evident that sufficient conditions for existence of an LSNE at the joint origin in higher dimensions can be obtained using standard results on the determinants, $\{\det(C_j)\}$, and traces, $\{\text{trace}(C_j)\}$, of the characteristic matrices.

Notice that the case with two parties of equal valence immediately gives a situation with $2\beta[1 - 2\rho_1]\sigma^2 = 0$, irrespective of the other parameters. However, if $\lambda_2 > \lambda_1$, then the joint origin may fail to be an LNE if $\beta \sigma^2$ is sufficiently large.