8. The Heart of a Polity

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8.1 Rational Choice Models of Electoral Politics

Arrovian classification results applied to voting theory have created a serious obstacle to constructing a rational choice theory of representative politics. A natural framework within which to work is one where the electorate has preferences that can be described in terms of a domain X, called the policy space, representing the possible creation and distribution of public goods in the political economy. In the most general context X would be a manifold described by a transformation possibility frontier between the private goods economy and the "public goods" economy.

Significant work has been done (Denzau and Parks 1983) to show the existence of a joint political economy equilibrium. Here the task was to show that the conditions for a general equilibrium in the economic domain were compatible with existence of a political equilibrium in the polity. It was known that as long as the political domain, X, was one-dimensional, then there would exist a political equilibrium (at the "median" voter position). If all political competition were fundamentally one-dimensional (based on a left-right dimension, say, or characterized by tax-rates) then a combined political economy equilibrium would exist. Moreover the political equilibrium could be found by "binary Downsian competition" between two office-seeking parties competing against each other to secure a majority of the votes (Downs 1957).

Unfortunately for this model, results by Plott (1967), Kramer (1973), McKelvey (1976), Schofield (1978, 1983) and finally McKelvey and Schofield (1986, 1987) showed that any democratic voting rule is generically unstable as long as the dimension is sufficiently high. "Generically" refers to an open dense set of profiles in a topological space of all smooth, utility profiles on the manifold X (full definitions of this term and related concepts can be found in Section 8.2). For example if majority rule is used, and the society has an odd number of voters, and if an equilibrium (or majority "core") exists in two dimensions, then it will disappear under the slightest perturbation of the utility profile. In three dimensions endless cycling both within and outside the Pareto set is almost always possible. Initially it was thought that the result was an artefact of the mathematical model used, in the sense that "generic" referred to utilities rather than preferences, and that the instability resulted from the introduction of non-convex perturbations. However this "voting paradox" has been shown to be "generic" in a topological space of smooth preferences, and even of convex preferences, when X is a vector space (Schofield 1995a).

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These results are, however, only applicable to models of *direct democracy* or "committees," where no political representation is involved. We cannot draw any immediate inferences about the behaviour of representative democracies in general.

Over two decades of research on studying rational choice models of representative democracy have given us some understanding of the relationship between the type of assumptions made and the possibility of political equilibrium. Results are somewhat different depending on whether or not electoral risk is assumed. If there are two political agents simply competing for votes in a situation where they are unsure of the electoral response, then there will generally be a Nash equilibrium in the sense that there are "best" positions to pick that maximize the candidates' expected vote totals, given their beliefs (see Hinich and Ordeshook 1970; Enelow and Hinich 1984; Coughlin 1992 for full details of the "binary Downsian model of electoral politics"). Even if these agents have policy preferences, then one might expect an equilibrium (Cox 1984a). However, in the second case there is a problem of "credible commitment" (see Section 8.4). Why should voters trust these political agents?

In response perhaps to experimental evidence, reviewed in McKelvey and Ordeshook (1990), that committee voting is not unstable in two dimensions, considerable research effort has been devoted to the notion of the *uncovered set* (Miller 1980; McKelvey 1986). This set always belongs to the Pareto set of the voters, and converges to the voting core (if one exists). See Cox (1987) and the discussion in Section 8.2. Two justifications for the uncovered set have been given: (*a*) if two vote-maximizing parties compete against each other, then arguments can be made why they would converge to the uncovered set; (*b*) in large "committees" such as Congress, where voting is not entirely governed by party discipline, certain kinds of sophisticated voting behavior by representatives in response to "amendment procedures" can lead into the uncovered set (Shepsle and Weingast 1984). With respect to (*b*) it has been observed (Ordeshook and Schwartz 1987) that most amendment procedures (particularly those generally used in Congress) do not have the required property. This observation has somewhat weakened the justification for using the uncovered set as an equilibrium notion.

My intention here is to present an outline of an integrated theory of multiparty competition (where "multi" means at least three). Unfortunately the uncovered set cannot be used directly since this set depends for its definition on the underlying preference relation obtained from the voting rule. Since coalitions are the "agents" in a multiparty system, an appropriate equilibrium notion must be based on behavior by coalitions. In Section 8.2 I define an equilibrium notion called the *heart*. It is Paretian and satisfies a continuity property (called lower hemi-continuity) that implies convergence to the core. Section 8.3 gives a number of illustrations and empirical examples of the heart, to justify the argument that this notion captures the essence of political bargaining in a situation where parties are concerned about policy.

Section 8.4 reviews models of two party competition and argue that a direct generalization to a vote or seat-maximization model is implausible in the multi-party case. Instead I argue that a natural model is one where parties compete with each other by presenting *manifestos* to the electorate. These manifestos have a dual purpose. In the period prior to the election, they give the electorate a basis for evaluating the parties. I assume implictly that each profile of manifestos ($z = (z_1, \ldots, z_n)$ in the case of n parties) gives rise to a response by the electorate that can be described in one of two ways. If there is no electoral uncertainty, then the "electoral map" gives an allocation $e(z) = (e_1(z), \ldots, e_n(z))$, specifying the shares of the seats to the parties. This generates a "political heart", written $\mathcal{HD}(z)$, defined by the set of winning coalitions. (That is, $\mathcal{D}(z)$ consists of those coalitions that win once the seat shares, $e(z)$, are known.) After the election, bargaining then takes place between the parties within the heart. Section 8.5 analyzes a particularly simple version of this model, when there are only three parties and $\mathcal{D}(z)$ is fixed. In this simple model, the parties need not consider electoral response in choosing their manifestos. However, the model assumes that they will be committed to these declarations and analyses the relationship between their choices of manifestos and their likely coalition partners. The probabilities that various coalitions form are described by a vector, $\rho(z)$, defined by the profile of declarations. Schofield and Parks (1993) have shown that there is a pure strategy Nash equilibrium in this game. The expectation of the equilibrium outcome, across the three possible coalition governments, belongs to the Pareto set of the three parties. Finally, Section 8.6 suggests how to incorporate the electoral connection. With electoral risk, the response of the electorate is described by a list of probabilities $\xi(z) = (\ldots, \xi_t(z)), \ldots)$, where $\xi_t(z)$ is the probability that a coalition structure, $\mathcal{D}_t$, occurs, given the declaration, z, of manifestos. The assumption here is that the number of seats won by each party is less important than the post-election set of winning coalitions. Again the parties assume that they are committed to their declarations. With a joint choice, z, each party calculates the probability $\xi_t(z)$ that $\mathcal{D}_t$ occurs, and evaluates this state of the world using a "policy" selection g_t of the heart. More generally, the overall outcome $g(z)$ is described by a game form g_ξ which specifies how coalitions form and what perquisites are allocated to members. A von Neumann-Morgenstern utility function is used by each party to evaluate z.

The notion of a Nash equilibrium for such a game can be developed. It is conjectured that, under general conditions, a pure strategy Nash equilibrium exists. It is hoped that later research will substantiate this conjecture and demonstrate the nature of the relationship between the political equilibrium and the electoral distribution of preferences.

8.2 The Voting Paradox, the Uncovered Set, and the Heart in a Committee

Arrow's Theorem (1951) and later research showed that *any* social choice mechanism, σ, can be "badly behaved" whenever it satisfies minimal democratic principles. The result should, properly speaking, be viewed as a classification theorem, since the large class of results that have been obtained make fairly clear the relationship between the assumptions on σ and its properties. To some extent the theorem creates formidable theoretical difficulties for the development of a discipline of political economy founded on a rational choice model of human behavior. To understand these difficulties it is worthwhile reviewing the structure of the classification theorem.

The primitive in *discrete* social choice theory is an array or *profile* of individual preferences $P = (P_1, \ldots, P_n)$ on a set of alternatives, X. Social preference $\sigma(P)$ is simply a social ordering determined by P and the normative properties imposed on σ. A key property of $\sigma(P)$ is that there exists some non-empty choice set, or "core," $E(\sigma(P))$, of outcomes unbeaten under $\sigma(P)$. Moreover the choice should belong to the Pareto set of the preference profile. If P_N is the unanimity preference relation, then $E(P_N)$ stands for the Pareto set. A sufficient condition for a non-empty choice set is that $\sigma(P)$ always be acyclic. That is, it is impossible for there to exist a cycle of the form $x_1\sigma(P)x_2\sigma(P)\ldots x_n\sigma(P)x_1$. Any social choice mechanism σ gives rise to a class of coalitions, called $\mathcal{D}_\sigma$, that are decisive under σ. In other words, if all members of a decisive coalition agree, then it can control an outcome. $\mathcal{D}_\sigma$ is called *collegial* if some set of individuals belongs to every coalition in $\mathcal{D}_\sigma$. This set of individuals is called the *collegium* of $\mathcal{D}_\sigma$. An individual in the collegium is also called a *vetoer.* If no vetoer exists, then $\mathcal{D}_\sigma$ is called non-collegial. When $\mathcal{D}_\sigma$ is non-collegial, it is *classified* by an integer $k = k(\sigma)$ called the Nakamura number (1979). This integer is simply the cardinality of the smallest subfamily of $\mathcal{D}_\sigma$ which is non-collegial. In other words if we take $(k - 1)$ different coalitions from $\mathcal{D}_\sigma$, then they have a member in common. (For majority rule, in general, $k = 3$. However for the $\frac{3}{4}$-rule, k is clearly 4.)

If σ has a non-collegial family $\mathcal{D}_\sigma$ of decisive coalitions, then it is always possible to find a profile P such that $\sigma(P)$ is cyclic. While this does not necessarily imply that the choice set $E(\sigma(P))$ is empty, it does mean that there is no general reason for presuming that the choice exists. A further point to note is that any cycle that exists will involve at least k different alternatives. For example if σ is a majority voting rule, then the Nakamura number is generally three. Thus a voting rule will, in general, be well-behaved if (and only if, in some sense) there are not more than two alternatives.

For the case of a voting rule σ, defined entirely by a class of coalitions $\mathcal{D}$, we now write $P_\mathcal{D}$ for the social preference induced by σ on the profile P. It is notationally clearer to regard $P_\mathcal{D}$ as a function $P_\mathcal{D} : X \longrightarrow 2(X)$ from X to the power set $2(X)$ (*i.e.,* all subsets of X). Thus $y \in P_\mathcal{D}(x)$ iff there exists some $M \in \mathcal{D}$ such that y is preferred to x by all members of M. Another way of writing this is that $P_\mathcal{D} = \cup \{P_M : M \in \mathcal{D}\}$ where P_M is the preference for coalition M. We write $k(\mathcal{D})$ for the Nakamura number of the rule, $\mathcal{D}$. The set $E(P_\mathcal{D})$ is usually called the *core* of the voting rule $\mathcal{D}$ at the profile P.

The first classification result states that if there are at least $k(\mathcal{D})$ alternatives, then it is possible to construct a profile, P, on these alternatives such that $E(P_\mathcal{D})$ is empty. Attempts to examine the frequency of the voting paradox were obliged to make ad-hoc assumptions on the relative frequencies of different preferences. The Arrovian classification theorem in this form does not give cause for concern for the development of a formal political economy, since it could easily be the case that no real society exhibits the particular preference profile required to construct social preference cycles. However one aspect of the theorem due to Gibbard (1973) and Satterthwaite (1975) suggested that social choice mechanisms would either be dictatorial or "manipulable" if preferences were "rich enough". The natural question then was: how rich does the domain of preferences have to be for Arrow's theorem to be relevant?

To put some structure on preferences it is natural to let X be a convex topological (vector) space. In this case we may require continuity and convexity properties on preference, before determining whether well-behaved social preference can be constructed. In this "category" of continuous convex preference, an Arrovian classification result again obtains. In this case the voting rule $\mathcal{D}$ will always exhibit an equilibrium, $E(P_{\mathcal{D}})$, as long as the dimension of the space, X, is no greater than $k(\mathcal{D})-2$ (see Greenberg 1979; Schofield 1984; Strnad 1985). On the other hand if the dimension of X is at least $k(\mathcal{D})-1$ then a continuous, convex profile, P, can be found such that $E(P_{\mathcal{D}})$ is empty and $P_{\mathcal{D}}$ cycles. Again this is a possibility theorem–a profile P can be *found* which generates cycles.

Rubinstein (1979), Cox (1984b), and Le Breton (1987) all attempted to answer the question "how often will $P_{\mathcal{D}}$ cycle"? A natural way to answer this question is to put a topology on the set $P(X)^N$ of profiles $P=\{P_1,\ldots,P_n\}$ for the society N. This can be done by considering the graph of a preference P_i and defining an ϵ−neighborhood, $V_\epsilon(P_i)$, to consist of all preferences whose graphs are within ϵ (in the so-called Hausdorff metric) of the graph of P_i. The set of preference profiles $P(X)^N$ is then given the product topology. Call this the *Hausdorff* topology on preference profiles.

A set V in $P(X)^N$ is *dense* iff for any profile, P, not in V, then any open neighborhood of P intersects V. In general a property ψ of profiles is *generic* with respect to a topology if it is true for an open dense set of profiles (in this topology). Rubinstein (1979), Cox (1984b), and Le Breton (1987) have shown essentially that the property $\psi=\{E(P_{\mathcal{D}})=\emptyset\}$ is *generic* in this topological space $P(X)^N$, endowed with the Hausdorff topology.

Notice that there is no dimension constraint in this result. The apparent contradiction between the results of Greenberg, *et al.* and Rubinstein, *et al.* is because of the possibility of non-convex preference. Even though P is a convex profile P, exhibiting a choice $E(P_{\mathcal{D}})$, there will exist a profile P' in any ε−neighborhood of P in the Hausdorff topology, but involving non-convex preferences, such that $E(P'_{\mathcal{D}})$ is empty.

The problem with the Hausdorff topology is that it is coarse in the sense that its open sets are "large". It is possible that, for a finer topology on $P(X)^N$, with more open sets, the set of profiles $\{P : E(P_{\mathcal{D}})=\emptyset\}$ would not be open-dense. Then the property $\psi=\{E(P_{\mathcal{D}})=\emptyset\}$ need not be generic with respect to the finer topology.

Results relevant to this concern were obtained by McKelvey (1976), Schofield (1978), and McKelvey and Schofield (1986, 1987). Consider now the set of smooth utility profiles, $U(W)^N$, endowed with the Whitney topology. In this topology two utilities are close if their values and all gradients are close. Then the instability aspect of the *classification theorem*, is that for any non-collegial rule $\mathcal{D}$, there exists an integer $w(\mathcal{D})$ satisfying $k(\mathcal{D}) - 1 \leq w(\mathcal{D})$, which also classifies σ and $\mathcal{D}$ in the following sense. If the dimension of X (namely $dim(X)$) is at least $w(\mathcal{D})$, then the property $\{E(P_{\mathcal{D}}(u)) = \emptyset\}$ is *generic*, in the Whitney topology $U(W)^N$. Here $P_{\mathcal{D}}(u)$ is the preference relation defined by the rule, $\mathcal{D}$, using the preference profile $P(u)$ induced, in the obvious way, from the utility profile, u. Properly speaking, $dim(X) \geq w(\mathcal{D})$ is sufficient for generic emptiness of the core in the interior of X, while $dim(X) \geq w(\mathcal{D}) + 1$ guarantees generic emptiness of the core in the boundary of X. Moreover, the property that cycles go almost everywhere in X is also generic if the dimension is at least $w(\mathcal{D}) + 1$. These two properties are together called the *generic voting paradox* for smooth utilities. Note that if $\mathcal{D}$ is majority rule, then $w(\mathcal{D}) = 2$ or 3 depending on whether n is odd or even (Schofield 1983). Saari (1995) has recently obtained a tight bound for $w(\mathcal{D}_q)$ in the case of a $q-$rule, $\mathcal{D}_q$, where $\mathcal{D}_q$ contains all coalitions of size q and above. In particular, he shows that if $dim\ (X) \leq w(\mathcal{D}_q) - 1$, then $\{u : E(P_{\mathcal{D}}(u)) \neq \emptyset\}$ contains an open set in $U(W)^N$. These results on the generic voting paradox for utilities imply a similar result for smooth preferences. Let $W(X)^N$ be the space of all smooth utility profiles whose components have isolated, non-degenerate critical points. Then there is a natural operation $P : W(X)^N \longrightarrow \mathcal{T}(X)^N \subset P(X)^N$, where $\mathcal{T}(X)^N$ is the space of preferences obtained from such utilities (endowed with a topology induced from $U(X)^N$). As before if u is a utility profile, then $P(u)$ is the appropriate preference profile. It has recently been shown (Schofield 1995a,b) that the classification result involving $w(\mathcal{D})$ is generic in the topological space $\mathcal{T}(X)^N$. In other words the set of smooth preference profiles

$$\{P(u) \in \mathcal{T}(X)^N : E(P_{\mathcal{D}}(u)) = \emptyset\}$$

is open dense as long as $dim(X) \geq w(\mathcal{D})$. Moreover, generically, $P_{\mathcal{D}}(u)$-cycles can go almost anywhere. An important feature of this result is that the topology used on $\mathcal{T}(X)^N$ makes use of information encoded in the gradients of the underlying utility representation. For this reason we call this the C^1-topology on $\mathcal{T}(X)^N$ and write $\mathcal{T}_1(X)^N$ when this topology is used. "C^1" refers to "continuous first differentials". If gradient information is not used, then the topology is based on the C^O-topology for smooth utilities, and in this case we write $\mathcal{T}_O(X)^N$. Again C^O means continuous, but without reference to the values of differentials. The Hausdorff topology, discussed previously, is equivalent to the use of $\mathcal{T}_O(X)^N$. It can be shown that the Rubinstein-Cox-Le Breton result is not true in the "finer" topology $\mathcal{T}_1(X)^N$, without a dimension constraint (Schofield 1995a). One case that is useful to analyse is when preferences are Euclidean. In this case the two topologies, $\mathcal{T}_O(X)^N$ and $\mathcal{T}_1(X)^N$ are identical. We say a preference P_i is *Euclidean* on X (where the vector space, X, is endowed with a Euclidean metric $||\ ||$) iff there exists some "bliss" point x_i such that

$$y \in P_i(x) \text{ iff } \| y - x_i \| < \| x - x_i \| .$$

Such preferences are used frequently in spatial voting theory because they have a simple geometric structure. Indeed there is a more profound reason for considering preferences of this kind. If P_i has a bliss point at x_i, then any utility function u_i which represents the preference P_i has a gradient $du_i(x)$ at x which is a function of the vector $(x_i - x)$. Consequently if u_i and u_i' are "Euclidean" utility functions which have bliss points, x_i and x_i', which are close, then u_i and u_i' will also be close in the C^1-topology. Thus the C^0- and C^1–topologies are identical (homeomorphic) in the case of Euclidean preferences.

Consequently, the space of Euclidean preferences with the C^1-topology can be identified with the product space $X^N = \Pi_{i=1}^n X_i$, where each X_i is a copy of X and X^N has the Euclidean topology. It follows, from the generic voting paradox for utilities, that in a topolgical space of Euclidean preferences, $\{P \in X^N : E(P_\mathcal{D}) = \emptyset\}$ is open dense as long as $dim(X) \geq w(\mathcal{D})$, for $\mathcal{D}$ non-collegial. In other words, the classification of $P_\mathcal{D}$ for smooth utilities implies a similar classification for Euclidean preferences. As a consequence, the space, X^N, of such profiles can be viewed as a *model* for $\mathcal{T}(X)^N$.

A corollary of this is that if it is possible to "solve" the generic voting paradox in X^N, then it is possible to solve it in $U(X)^N$. In light of the above discussion, the Rubinstein–Cox–Le Breton result is not valid for Euclidean preferences, without the dimension constraint given by $w(\mathcal{D})$. Consequently the voting paradox does not occur for the rule, $\mathcal{D}$, in dimension below $k(\mathcal{D})-2$. The problem only becomes relevant in dimension $w(\mathcal{D})$ or above.

In an effort to solve the voting paradox, a number of authors have developed the notion of the *uncovered set* (Miller 1980; Shepsle and Weingast 1984; McKelvey 1986). Let $P : X \longrightarrow 2(X)$ be a preference correspondence on X, representing the social preference. Define the *covering correspondence*, $\overline{P}$ of P by $\overline{P} : X \longrightarrow 2(X)$ where $y \in \overline{P}(x)$ iff $y \in P(x)$ and $P(y) \subset P(x)$. The uncovered set, $\overline{E}(P)$, of the correspondence P is $\overline{E}(P) = \{x \in X : \overline{P}(x) = \emptyset\}$. Because of the set inclusion, $\overline{P}$ will be acyclic. If X is compact and $\overline{P}$ is "continuous" then $\overline{E}(P)$ will be non-empty. (Bordes, *et al.* 1992).

Cox (1987) has shown that the uncovered set satisfies three crucial properties. Suppose $\mathcal{D}$ is the majority voting rule with n odd. Let P be a Euclidean preference profile (actually the following result is valid if each P_i is obtained from "quasi-concave" utility functions, u_i with single bliss point x_i).

The uncovered set at the profile P, for majority rule, $\mathcal{D}$, is $\overline{E}_\mathcal{D}(P) = E(\overline{P}_\mathcal{D}) = \{x \in X : \overline{P}_\mathcal{D}(x) = \emptyset\}$. Then $\overline{E}_\mathcal{D}$ is

1. non-empty and "Paretian" : $\emptyset \neq \overline{E}_\mathcal{D}(P) \subset E(P_N)$;

2. "core-like" : if $E(P_\mathcal{D}) \neq \emptyset$, then $\overline{E}_\mathcal{D}(P) = E(P_\mathcal{D})$;

3. "convergent to the core" : if $P \longrightarrow P'$ and $E(P'_\mathcal{D}) \neq \emptyset$ then $\lim_{P \to P'} \overline{E}_\mathcal{D}(P) = E(P'_\mathcal{D})$.

Actually Cox proved a weaker version of (3), namely convergence (in terms of bliss points) to a point core. The convergence property (3) would follow from (2) and the "lower hemi-continuity" of the correspondence $\overline{E}_{\mathcal{D}}$.

A correspondence $\mathcal{H}$ between two topological spaces X, Y is lower hemi-continuous (*lhc*) iff for any open set V in Y, the set $\{x \in X : \mathcal{H}(x) \cap V \neq \emptyset\}$ is open in X. From *lhc*, if $x_n \longrightarrow x$ in X, then $\mathcal{H}(x_n) \longrightarrow \mathcal{H}(x)$ in Y. I now wish to propose an alternative equilibrium notion to $\overline{E}_{\mathcal{D}}$ which satisfies (1), (2) and *lhc* (and thus (3)). Although the definition is given in terms of Euclidean preferences, all the results are valid for more general "convex" preferences derived from smooth quasi-concave utilities (see Schofield 1995b). Moreover the definitions and results are valid for any voting rule, $\mathcal{D}$.

The key to constructing the covering correspondence $\overline{P}$ is that $\overline{P}$ is a sub-correspondence or "reduction" of P in the sense that $y \in \overline{P}(x) \Longrightarrow y \in P(x)$. However if $y \in P(x)$ but $P(y) \not\subset P(x)$ then $y \notin \overline{P}(x)$.

We proceed in a similar way to reduce $P_{\mathcal{D}}(x)$. Note that $y \in P_{\mathcal{D}}(x)$ iff $y \in P_M(x)$ for some M in $\mathcal{D}$. So $P_{\mathcal{D}} = \cup \{P_M : M \in \mathcal{D}\}$. Consider a single coalition M. It is impossible to reduce P_M in the above way because if $y \in P_M(x)$ and $z \in P_M(y)$, then by transitivity of individual preference, $z \in P_M(x)$. So $\overline{P}_M = P_M$. However we can construct a reduction $\tilde{P}_M$ of P_M called the *efficient* preference for the coalition. Suppose that $y \in P_M(x)$ for some y, x. Suppose further that there exists no point $z \in X$ with $z \in P_M(y)$ and $\| z - x \| = \| y - x \|$. Then define $y \in \tilde{P}_M(x)$. On the other hand if $\exists\, z \in X$, with $z \in P_M(y)$ and $\| z - x \| = \| y - x \|$ then $y \notin \tilde{P}_M(x)$.

Figure 1(a) illustrates the definition in the case when $M = \{1, 2\}$ and the individuals have Euclidean preferences. The spherical indifference curves for $\{1, 2\}$ through x are labelled I_1, I_2. Let $P = P_{\{1,2\}}$ and $\tilde{P} = \tilde{P}_{\{1,2\}}$. Now $y_3 \in P(x)$. However $y_2 \in P(y_3)$ and y_2, y_3 are on the sphere $S = S(x : \| y - x \|)$ with center x and radius $\| y - x \|$. Thus $y_3 \notin \tilde{P}(x)$. Notice that y_1, y_2 both lie in S. However 2 prefers y_2 to y_1 while 1 prefers y_1 to y_2. Clearly all points on the arc in S between y_1 and y_2 belong to $\tilde{P}(x)$.

The same argument holds in Figure 1(b). Here $y_3 \notin \tilde{P}(x)$ because $y_3 \notin P(x)$, while 1 and 2 disagree over their preferences for y_1 and y_2.

In Figure 1(a) the preference for y_3 over x is "inefficient" in some sense. A more abstract argument can be made for eliminating points such as y_3 from $P(x)$, but the formalism requires the idea of preference cones. See Austen-Smith (Chapter 9 of this volume) for the technical definitions. The general idea is simply that, since Euclidean preference is denominated in distance, it is appropriate for coalitions, choosing to move from x, to compare like with like, namely points such as $\{y_1, y_2, y_3\}$ equidistant from x. Now define

$$\tilde{P}_{\mathcal{D}} = \cup \{\tilde{P}_M : M \in \mathcal{D}\}$$

by $y \in \tilde{P}_{\mathcal{D}}(x)$ iff $y \in \tilde{P}_M(x)$ for some M belonging to $\mathcal{D}$. $\tilde{P}_{\mathcal{D}}$ is the *efficient* preference for the rule $\mathcal{D}$ at the profile P. It can be shown (Schofield 1995b) that $P_{\mathcal{D}}(x) = \emptyset$ iff $\tilde{P}_{\mathcal{D}}(x) = \emptyset$, for any "convex" preference profile, P. Thus $E(P_{\mathcal{D}}) = E(\tilde{P}_{\mathcal{D}})$. Consequently if the core for $P_{\mathcal{D}}$ exists then so does $E(\tilde{P}_{\mathcal{D}})$. What happens if $E(P_{\mathcal{D}}) = \emptyset$?

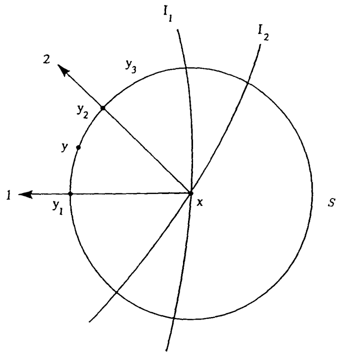

Figure 1(a)

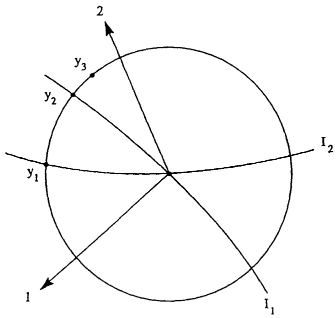

Figure 1(b)

Define $\Gamma(\tilde{P}_{\mathcal{D}})$ in the following way: $x \in \Gamma(\tilde{P}_{\mathcal{D}})$ iff $\exists$ at least three vectors $\{v_i : i = 1, 2, 3\}$ such that each $v_i \in \tilde{P}_{\mathcal{D}}(x)$ and

$$x = \sum_{i=1}^{n} \lambda_i v_i,$$

where $\lambda_i \geq 0$, but not all are zero.

From previous results (Schofield 1984) it is known that if $\Gamma(\tilde{P}_{\mathcal{D}}) = \emptyset$, then $E(\tilde{P}_{\mathcal{D}}) \neq \emptyset$, whenever X is a compact convex set. (In fact, for a Euclidean profile, the emptiness of Γ is a necessary and sufficient condition for non emptiness of E.) Now define the *heart*, $\mathcal{H}_{\mathcal{D}}$ for the profile, P, and rule, $\mathcal{D}$, by $\mathcal{H}_{\mathcal{D}}(P) = E(\tilde{P}_{\mathcal{D}}) \cup \text{clos}\, \Gamma(\tilde{P}_{\mathcal{D}})$.

(For technical reasons it is necessary to ensure $\mathcal{H}_{\mathcal{D}}(P)$ is closed, and clos just means take the closure.) It then follows (Schofield 1993a, 1995b) that $\mathcal{H}_{\mathcal{D}}$ is:

1. non-empty and Paretian : $\emptyset \neq \mathcal{H}_{\mathcal{D}}(P) \subset E(P_N)$;
2. core-like : if $E(P_{\mathcal{D}}) \neq \emptyset$ then $\mathcal{H}_{\mathcal{D}}(P) = E(P_{\mathcal{D}})$;
3. lower hemi-continuous.

If we identify a Euclidean preference profile P by the list of bliss points, $\{x_1, \ldots, x_n\}$ then $\mathcal{H}_{\mathcal{D}}$ is a closed correspondence $\mathcal{H}_{\mathcal{D}} : X^N \longrightarrow 2(X)$.

Because $\mathcal{H}_{\mathcal{D}}$ is lower hemi-continuous, if the sequence $\{P_k = (x_{1k}, \ldots, x_{nk})\}$ converges to $P' = (x'_1, \ldots, x'_n)$ and the profile P' has a core, then $\mathcal{H}_{\mathcal{D}}(P_k)$ also converges to this core.

It is known from previous results that cycles may occur under $\tilde{P}_{\mathcal{D}}$, but in a certain sense they can occur only within $\Gamma(\tilde{P}_{\mathcal{D}})$. If we modify $\tilde{P}_{\mathcal{D}}$ to a subpreference $P^*_{\mathcal{D}}$ of $\tilde{P}_{\mathcal{D}}$ in the following way, then no cycles can occur.

Define $P^*_{\mathcal{D}}$ by $y \in P^*_{\mathcal{D}}(x)$ iff $y \in \tilde{P}_{\mathcal{D}}(x)$ and $y \notin \mathcal{H}_{\mathcal{D}}(P)$. Then it can be shown that, for any point $x_0 \notin \mathcal{H}_{\mathcal{D}}(P)$ and any sequence $\{x_1, \ldots, x_r\}$ with $x_k \in P^*_{\mathcal{D}}(x_{k-1})$ for $k = 1, \ldots, r$, then the distance between x_r and $\mathcal{H}_{\mathcal{D}}(P)$ is strictly less than the distance from x_0 to $\mathcal{H}_{\mathcal{D}}(P)$. Thus $P^*_{\mathcal{D}}$ converges to the heart. That is to say, $\mathcal{H}_{\mathcal{D}}(P)$ is an "attractor" for the preference correspondence $P^*_{\mathcal{D}}$. (It should also be emphasized that it is assumed in this model that $P^*_{\mathcal{D}}$ is based on continuous "preference paths". See Schofield (1978) for the original definition.)

We have now constructed a preference correspondence $P^*_{\mathcal{D}}$ that is a subpreference of both the efficient preference $\tilde{P}_{\mathcal{D}}$ and the social preference $P_{\mathcal{D}}$. Further $P^*_{\mathcal{D}}$ exhibits no cycles and it converges to the heart, (a subset of the Pareto set). Indeed if the core is non-empty, then it converges to the core. The preference $P^*_{\mathcal{D}}$ has all the desirable properties of the covering correspondence but more importantly it is based on "efficient" behavior by coalitions.

I conjecture that the correspondence $\mathcal{P}^*_{\mathcal{D}}$ is a sub-correspondence of the covering correspondence, that is $P^*_{\mathcal{D}}(x) \subset \overline{P}_{\mathcal{D}}(x)$ for all x, (at least in the case of Euclidean preferences). More precisely if $x \notin \mathcal{H}_{\mathcal{D}}(P)$ so that $y \in P^*_{\mathcal{D}}(x)$ for some $y \in X$, then it is easy to show that for any neighborhood V of x there exists $y' \in V$ such that $y' \in P^*_{\mathcal{D}}(x)$. Moreover y' also covers x. Thus $P^*_{\mathcal{D}}$ is a "localization" of the covering relation. It appears therefore that $\mathcal{H}_{\mathcal{D}}(P)$ can be interpreted as the set of points that are locally uncovered. The advantage of this equilibrium notion is that determining whether a point lies in the heart or not is a local problem. Determining whether a point lies in the uncovered set is a global problem and almost impossible to solve when $n > 3$. It should be noted that the heart is formally defined for a situation where each individual, i, in the voting procedure has well-defined preferences on the space, X. This means that the concept is formally appropriate only for spatial models of direct democracy, or of a *committee*.

However it has been traditional in examining coalition bargaining in multiparty situations to suppose that parties do have well-defined policy preferences. Section 8.3 assumes that parties do indeed have "spatial" or policy preferences and uses the heart to analyse the outcomes of coalition negotiation.

8.3 The Heart in Multiparty Politics

To illustrate the concept of the heart, consider Figure 2 which presents estimates of party position of the five parties in Sweden in 1976 for a two-dimensional policy space, X. The spatial map is drawn from my own factor analysis of data made available by Holmstedt and Schou (1987). The left-right dimension concerns economic policy, while the north-south dimension is defined by non-economic welfare concerns. With 349 seats, a majority is 175. Because the space of Euclidean profiles is a model space, we can without great loss of generality assume each party has Euclidean preferences, based on a bliss point, as in Figure 2.

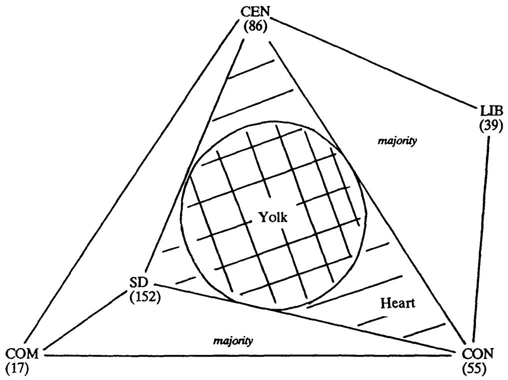

Figure 2. Sweden in 1976.

The Social Democrats (SD) with 152 seats face a bourgeois coalition of three parties (Center, Liberals and Conservatives) controlling 180 seats. We can represent the pattern of winning coalitions by three important majorities, namely $\{SD, CON\}, \{SD, CEN\}$, and $\{CEN, LIB, CON\}$. The two majorities involving the SD can be represented by "median" lines through the SD position, while the bourgeois majority is described by one median line, $\{CEN, CON\}$. The majorities they describe bound a triangular area (shaded in Figure 2) with vertices $\{SD, CEN, CON\}$. This area can easily be shown to be the heart of this game. It is also easy to see that if the party positions move continuously, then the heart will also change shape, continuously.

Table 1.
SEAT STRENGTHS IN SWEDEN

	Seats			Weight	
	1973	*1976*	*1979*	*1973*	*1976*
Communist Left (COM)	19	17	20	1	1
Social Democrats (SD)	156	152	154	5	4
Center Party (CEN)	90	86	64	2	2
Liberals (LIB)	34	39	38	2	2
Conservatives (CON)	51	55	73	2	2
Total	*350*	*349*	*349*	*7/12*	*6/11*

We can contrast this figure with the situation in 1973 where the SD had 156 seats and a majority of 176 was required (see Table 1 and Figure 3). In this case the median lines all pass through the SD position. That is, there is no majority coalition, M, such that the compromise set for M (namely the convex hull $\{x : i \in M\}$ of the bliss points) excludes the SD position. Consequently the core is precisely the SD position. By definition the core and the heart are identical. Given the information on party weights, and the estimates of party position, SD can be termed a "core party". Most importantly, this core property is "structurally stable" (Schofield 1986) since it is insensitive to (small) errors in the estimates of party positions.

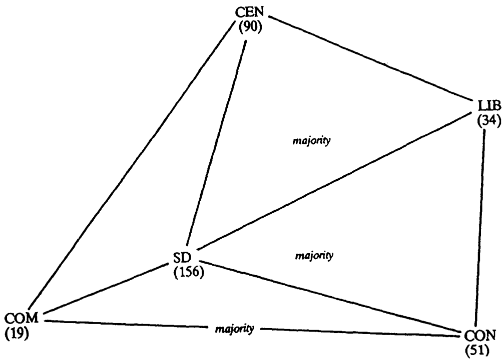

Figure 3. Sweden in 1973.

We may give some theoretical and substantive reasons why two dimensions appear important in coalition bargaining. In 1973 the SD formed a minority (non-majority) one-party government. This would be predicted by the fact that it is in the core position in the two-dimensional space. However a one-dimensional model (based on the economic dimension alone) would predict a two party coalition $\{SD, CEN\}$ since this two party group comprises the one-dimensional core. In fact in this period the SD had strongly committed the "country to a large scale expansion of nuclear generating capacity" (*Keesing Contemporary Archives* 28056), and the Center Party equally strongly objected. This cleavage was important in characterizing the second dimension. In 1976 the SD and Communist Left lost their blocking capability and the three-party bourgeois government formed, lasting (except for a minority liberal party government in 1978-79) until 1982. As we have emphasized, differences of policy opinions between SD and CEN made the bourgeois coalition likely once it obtained a majority. However this coalition was not *inevitable*. Between 1951 and 1957 the two party coalition, $\{SD, CEN\}$, did govern, even though the three-party bourgeois group controlled a majority after the election of 1956. A natural one-dimensional core model would have predicted a minority SD government up to 1956 and a minority Center Party government after 1956.

These observations imply that a model incorporating at least two dimensions is necessary for predicting government formation in Sweden. There is a further important inference. In a three-dimensional policy space it is "generically" the case that no party can occupy a core position (see Schofield 1989) unless it controls a majority of the seats. Yet in Sweden the SD party has on numerous occasions formed a minority government without control of a majority. Often, of course, the SD had the tacit support of the Communist Left, but this support could not always be assumed. This gives some theoretical justification for the inference that in Sweden the underlying policy space is neither one nor three-dimensional but two-dimensional.

If we accept the logic of this model, then it follows that a minority one-party government can only occur in two dimensions, in a stable fashion when a "dominant" party occupies the core position. Theoretical results (Schofield 1995c) have shown that any "smaller" party cannot occupy a "structurally stable" core position, in two dimensions. The terms "dominant" and "smaller" are defined technically, and refer not just to the number of seats, but to the minimum integer weights that specify the winning coalition structure. A voting rule $\mathcal{D}$ has an integer representation iff there is a representation of $\mathcal{D}$ by a quota, q, and a "profile" of minimum integer weights $\{q : w_1, \ldots, w_n\}$ such that $\sum_{i \in M} w_i \geq q$ iff $M \in \mathcal{D}$. Then say party i is "larger" than j (or j is "smaller" than i) iff $w_i > w_j$. Party i is dominant if $w_i > w_j$, for all $j \neq i$. See Table 1 for the "quotas" and minimum integer weights for the parties in 1973 and 1976. Clearly SD is "dominant" and the rest are "smaller" in both years.

However, as we noted previously, the coalition $\{COM, SD\}$ after the 1973 election is blocking (in the sense that it can deny the bourgeois group a majority) whereas in 1976 and 1979 it is not blocking. Schofield (1995c) introduced the term "strongly dominant" to characterize a party that was not only dominant but could be a member of such a two-party blocking coalition. Using results by McKelvey and Schofield (1987), it was shown that only a strongly dominant party, such as the SD in 1973, could stably occupy a core position in two dimensions. Since we know that the SD is not strongly dominant in 1976 and 1979, we can infer that it is impossible for it to occupy a core position in a stable fashion. It is this reasoning, I believe, that can provide an understanding of why relatively small shifts in election results can produce the big shift from a minority government to a majority coalition. Since a one-dimensional model always exhibits a core, it cannot account for a shift of this kind.

We can use Figure 2 to contrast the heart and the uncovered set. The easiest way to characterize the uncovered set is in terms of the *yolk*—the smallest ball that touches all median lines. In this case it is clear that the yolk is a ball located between the SD position and the $\{CEN, CON\}$ median in Figure 2. The uncovered set is known to lie inside a ball centered on the yolk, with radius no more than four times that of the yolk. Since the larger ball includes the entire Pareto set (the quadrilateral $\{COM, CEN, LIB, CON\}$), we only know the uncovered set lies inside the Pareto Set. The heart on the other hand is the triangle $\{SD, CEN, CON\}$. This triangle seems most relevant to the understanding of coalition bargaining in Sweden. Consequently there is an underlying prediction made by the heart that one of its three "bounding" coalitions will form. As we saw in the 1976 situation, these three bounding coalitions are $\{SD, CON\}, \{SD, CEN\}$ and $\{CEN, LIB, CON\}$. The heart is offered here as a general equilibrium concept suitable for analyzing coalition bargaining in multi-party systems.

The hypotheses implicit in this example can now be set out more formally. We use the terms minority for non-majority, minimal winning for majority with no surplus party, and surplus for majority but non-minimal. A "core party" is one that occupies the core of the spatial voting game generated by the various party weights and positions.

A. Theoretical Propositions:

1. In a one-dimensional policy space there will always exist a core party. This follows because $k(\mathcal{D}) \geq 3$ for any weighted majority rule. It is possible for the core party to be the smallest party.

2. In dimension $w(\mathcal{D}) - 1$, if there is a core party, then its position must belong to the core in any one-dimensional subspace. If a "smaller" party occupies the core in this dimension, then it is unstable with respect to perturbations in position. Consequently only a "strongly dominant" party (if one exists) can stably occupy the core position.

3. No "structurally stable" core is possible in dimension $w(\mathcal{D})$. Generically the heart will consist of either a star-shaped or convex set within the Pareto set.

4. In general $w(\mathcal{D}) \leq 3$ for weighted voting rules requiring a simple majority of the seats. However, if one party has a blocking majority of the seats, then $w(\mathcal{D}) = \infty$. (A blocking majority for a party is a sufficient number of seats so that no other group of parties controls a simple majority of the seats.)

5. With three parties, none of whom has a blocking majority, then $w(\mathcal{D}) = 2$. Moreover, with an odd number of parties, all with identical minimum integer weight, then $w(\mathcal{D}) = 2$.

6. If the core is empty then the dominant party will always occupy a position on the boundary of the heart. In particular in a situation where three parties all have equal minimum integer weight, so no one blocks, then, generically, all three are on the heart boundary.

B. Empirical Hypotheses:

7. Any government that forms will include the core party (if one exists). In particular if a minority one party government forms, then it will comprise the party at the core position.

8. If there is no core (and so at least two dimensions), then one of the coalitions bounding the heart will form. In particular a single party minority government is impossible.

C. Deduced Hypotheses:

9. If the policy space is one-dimensional then it is possible for a smaller party to form a minority government, excluding the dominant party, as long as the smaller party is at the core. In any case, every government includes the core party.

10. If the policy space is two dimensional, and a "stable" one-party minority government (with no other party support) occurs, then it must comprise the strongly dominant party.

11. If the policy space is three-dimensional, then a "stable" one-party minority government cannot occur, except when that party has a blocking majority.

D. Plausible Hypotheses:

12. If the policy space is two dimensional but the core is usually empty, then minority governments will be unusual. If $w(\mathcal{D}) = 2$, then minimal winning governments will be typical.

13. If the policy space is two dimensional and $w(\mathcal{D}) = 3$, but there is a dominant party nearly always at the core position, then that party will tend to "control" coalition politics.

14. If the policy space is two dimensional, and a "dominant" party usually occupies the core position, then, in specific instances when the core is empty, this "dominant" party will be excluded from government.

This is a complex set of hypotheses, incorporating both theoretical results on the core and heart, and some empirical hypotheses. The "plausible" hypotheses (12-14) do not follow directly, but they are consistent with the general model. These hypotheses can be evaluated in terms of the data presented in Table 2. These data are assembled from Schofield (1993b) and show coalition types for twelve post-1945 European polities. Column 1 in this table enumerates the number of single party minority governments that have formed. In brackets (in this column) is the number of single party minority governments comprising a "smaller" party.

In contradiction to hypothesis 10, in Denmark a smaller party, Venstre (Liberals) formed a one party minority government in 1947 and in 1973. In both cases however there was tacit support from right wing parties such as the Conservatives. Moreover in 1947 and 1975 the supported minority Venstre government gave way to a supported minority *SD* government. On the usual left-right economic dimension Venstre was not at a core position. It is likely that the situation was one of two dimensions with an empty core.

In Sweden the three party bourgeois coalition (Center, Conservatives and Liberals), described above, resigned in October 1978 as a result of serious disagreements on nuclear energy policy. The Liberals took over as a (very) minority goverment. After an election in September 1979 the three party coalition reformed. This event contradicts hypotheses (9) and (10), since the liberal party is usually assumed to be to the right of the median or core position on economic policy. It is possible however that the disagreements over nuclear policy were so extreme as to temporarily overwhelm the ability of the bourgeois parties to compromise. It is plausible therefore that the Liberals were at the median position on a nuclear policy dimension, positioned between the Social Democrats on one side and the Center and Conservative parties on the other.

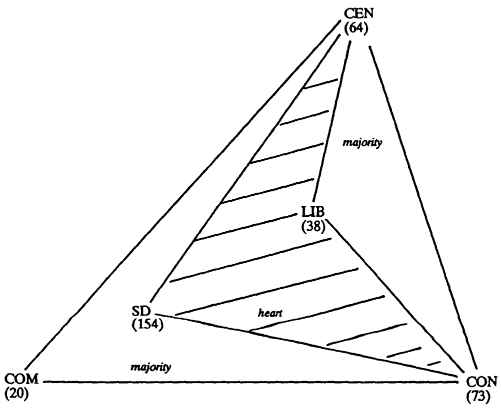

Figure 4. Sweden in 1979.

Figure 4 presents estimates of the positions of the parties based on their manifestos during the 1979 election. It is evident that *LIB* shows a change in position from 1976 to one nearer *SD*. The heart has changed shape from 1976 (it is shown shaded again in Figure 4). In fact a possible inference from this Figure is that the Liberals were considering an alliance with *SD* in the period 1978 to 1979.

The data on single party governments in Table 2 does seem to give quite strong confirmation that two dimensions are relevant, particularly in the Scandinavian countries. No confirmation is provided that only one-dimension is relevant.

In empirical work reported in Schofield (1993b), factor analysis of party manifestos based on data collected by Budge, *et al.* (1987) was used to estimate party positions. In Austria, Germany, and Luxembourg the party system is such that $w(\mathcal{D}) = 2$, so core positions cannot be expected in non-majority situations. Table 2 shows that single party minority governments are rare in the first six countries listed in the table. Notice that minimal winning coalitions are very frequent in these countries.

Table 2.
COALITION TYPES IN EUROPE

	Single Party Min.	*Other Min.*	*Minimal Winning*	*Surplus*	*Maj.*	*Total*
Austria		1	6		6	13
Belgium	1(0)	1	15	4	1	22
Germany			10		2	12
Luxembourg			9	1		10
Iceland	1(0)	1	10	2		14
Netherlands		3	6	8		17
Denmark	12(2)	6	2			19
Norway	7(0)	1	3		4	15
Sweden	9(1)	1	5		1	16
Finland	4(0)	6	5	17		32
Ireland	4(0)	1	3		4	12
Italy	7(0)	7	3	18		35

(Taken from Schofield, 1993b.)

In Belgium and the Netherlands a Center party tends to "dominate" politics. In the Netherlands the Christian People's Party (KVP) or its successor, the Christian Democratic Appeal (CDA), are estimated to be at the core position after six of the eleven elections for which manifesto data (Dittrich 1987) are available. Either the CDA or KVP has been in every government. This gives some confirmation for hypothesis (13). It is possible that there is, in fact, only one relevant dimension in the Netherlands, and the KVP (CDA) occupies the core position. The non-occurrence of single party minority government suggests however that two dimensions are important.

Hypothesis (14) is intended to capture the essence of the political game in the Scandinavian countries. As we have seen in Sweden a left-wing Social Democrat Party faces a bourgeois coalition. When the bourgeois coalition fails to attain a majority, then an SD minority (or majority) government typically forms. This phenomenon also seems to characterize both Denmark and Norway.

In Belgium the situation is somewhat more complex. Before 1961 the instability integer, $w(\mathcal{D})$, is two, and as one would expect, two party minimal winning coalitions tend to form. After 1961, increasing complexity in coalition politics meant that the Christian Social Party (CS) tended to occupy a core position. It has been in every coalition government since then. As Table 2 shows, coalitions tend to be minimal winning or surplus. In some sense we can argue that the $KVP(CDA)$ in the Netherlands and the CS in Belgium dictate the nature of coalition politics. It is interesting that one party minority governments based on these core parties are rare in these two countries. Dodd's (1976) arguments on conflict cleavage might suggest that conflict between the left and right in the Scandianvian countries creates the context for minority one-party governments, whereas in Belgium and the Netherlands, a large centrist party is able to provide the context for coalition negotiation.

The situation in Ireland is extremely interesting. Over the years Fine Gael has gained strength over Fianna Fail. Until 1982 Fianna Fail governments were common. As discussed in Laver and Schofield (1990: Chapter 1) the results of the March 1987 election resulted in a hung Dail. Fianna Fail had 81 seats out of 166 but wished to form a minority one party government. Two independents voted for Fianna Fail, and at the election for Taoiseach, a third independent, Gregory, abstained, giving Fianna Fail a blocking majority and thus control of government. On a one-dimensional model Fianna Fail would be placed at the core position, and thus the "minority" government just described might be expected. However earlier in 1981 a minority two party coalition of Fine Gael and Labor formed against Fianna Fail. It would seem logical, as a consequence, to infer that there is more than one dimension in Irish politics, and that the relevant second dimension in some sense characterizes an extreme form of political antagonism between the two major parties. In fact the factor analysis by Mair (1987) does suggest that a second dimension (interpretable as "Irish Unity") separates the two major parties. With three significant parties one might expect $w(\mathcal{D})$ to be 2. However the large number of independents in the Dail makes it possible for one of the two larger parties to form a minority government, particularly if they have support of a small group of independents such as the Farmers. A reasonable inference, then, is that Irish politics is compatible with the two dimensional model of the heart presented here.

Italy is even more interesting. Until the last election in 1992 the Christian Democrat Party (*DC*) occupied the core position on both dimensions (Schofield 1993b), these being left-right (economic) and north-south (technology versus social harmony: see Mastropaolo and Slater 1987). As Table 2 illustrates every imaginable kind of coalition has occurred, all including the *DC*. It now appears that because of its political dominance, the *DC* was able to use its strategic core position to obtain economic resources so as to maintain its political power. The fragmentation that occurred in the 1992 election brought new parties into being (the Northern League, Anti-Mafia group, *etc.*,) and split the Communist Party. As a result the *DC* was, for the first time since the 1940's, no longer in control of the core position. (See Figure 11 in Schofield, 1993b.) Since 1993 of course Italian politics (or at least the names of the parties) have been dramatically transformed. It no longer seems to be the case that even the large party, Forza Italia, is in the core position (see Schofield 1995c for a brief discussion).

In this analysis, I have emphasized that a theoretical model based on the existence, or otherwise, of a core position is important for interpreting multiparty coalition politics. I have assumed in this section that the positions and political strengths of the parties are known (they are common knowledge) and that they effect coalition bargaining in the manner described. Moreover, throughout this description, I have focussed on the political control that a party has if it occupies a core position. In the Swedish example it was argued that the *SD* would be able to implement a minority government if and only if it happened to occupy the core position. If the political rewards from occupying such a position are high, then one might expect parties to attempt to position themselves at the core. It is this possibility of maneuvering in policy space that we shall explore in the next section. Moreover since we have based the analysis on spatial maps derived from manifesto data, we must present a model of how parties choose these manifestos.

8.4 Models of Party Competition and Electoral Politics

Section 8.1 referred to the "binary Downsian model of electoral politics" where two candidates (or parties) compete against each other in order to win office. Neither is interested in policy *per se*, but each chooses a policy and promises the electorate to implement it, if elected. The Classification Theorem governs the behavior of such a model. With a finite electorate and using majority rule, $\mathcal{D}$, we know $w(\mathcal{D}) = 2$ or 3 depending on whether the size of the electorate is odd or even. If there is more than one dimension to policy, there is no reason to expect a Nash equilibrium in this political game.

As we have noted, the uncovered set has been proposed as a choice concept applicable to such a game. If $y \in \overline{P}_{\mathcal{D}}(x)$ then any point z that beats y also beats x. In some sense the point x is "weakly dominated" by z. Nonetheless in general there is no "pure strategy" Nash equilibrium.

It is worth reviewing the general class of models of electoral competition to see how they predict candidate choice under *risk*. Suppose therefore that candidates 1, 2 choose "declarations" or "manifestos" z_1, z_2 ($\in X$) respectively to present to the electorate. Suppose further that it is impossible to say definitely whether z_1 beats z_2 or vice versa. "Risk" means that there are "common knowledge" probability functions $\xi_1, \xi_2 : X \times X \longrightarrow \Re$ where $\xi_i(z_1, z_2)$ is the probability that i wins when 1, 2 pick $\{z_1, z_2\}$ respectively. The outcome is a finite lottery $g(z_1, z_2) = \{(z_1, \xi_1(z_1, z_2)), (z_2, \xi_2(z_1, z_2))\}$ which we can regard as a point in $\tilde{X}$, the set of all finite lotteries on X. The expectation is $\underline{Ex}[g(z_1, z_2)] = \xi_1(z_1, z_2)\, z_1 + \xi_2(z_1, z_2) z_2 \in X$. We can also include the possibility of a tie, by supposing that, with probability $1 - \xi_1 - \xi_2$, both candidates obtain the same number of votes. Then each party can use a von Neumann-Morgenstern utility function, U_i, to evaluate $g(z_1, z_2)$. That is

$$U_i(g(z_1, z_2)) = \xi_1(z_1, z_2)\, u_{i1}(z_1, z_2) + \xi_2(z_1, z_2) u_{i2}(z_1, z_2) + (1 - \xi_1 - \xi_2) u_{i3}(z_1, z_2).$$

Eq. 1

In the Downsian case, the candidates (or parties) are concerned simply to win. In this case we can assume that party 1 receives 1, say, if it wins the election (*i.e.*, $u_{11}(z_1, z_2) = 1$) whereas it receives -1 if it loses ($u_{12}(z_1, z_2) = -1$). For a draw, $u_{13} = 0$. Under certain conditions there will exist a Nash equilibrium. More formally, the strategy space for party i is a copy X_i of X.

The *ith* best response correspondence is

$$R(U_i) \; : \; X_j \; \longrightarrow \; 2(X_i)$$

where $R(U_i)(z_j) \; = \; \arg\max_{z_i \in X} \; \{U_i(g(-, z_j))\}$.

This just means i chooses z_i to maximize $U_i(g(,))$ given that the opponent plays z_j.

The joint best response correspondence is

$$R(U_1, U_2) : \; X_1 \; \times \; X_2 \; \longrightarrow \; 2(X_1 \; \times \; X_2)$$

where each plays best response to the other. A pure strategy *Nash equilibrium* ($PSNE$) is a fixed point of the correspondence $R(U_1, U_2)$.

In case there is no $PSNE$, there will generally be a mixed strategy Nash equilibrium ($MSNE$) where each party plays a mixed strategy in $\tilde{X}$ (see Kramer 1978).

In the case of deterministic voters, the win functions ξ_1, ξ_2 will take values of $0, 1$ or $\frac{1}{2}$ (in the case of a draw). It is important to note that this case can be viewed as a degenerate version of the probabilistic model. However in the absence of a core point in the distribution of voter preferences, there will be no $PSNE$ in the candidate game with deterministic voters. Moreover the "win" functions $\{\xi_1, \xi_2\}$ will be discontinuous in the candidate strategies $\{z_1, z_2\}$. For this reason the uncovered set has been proposed as a "solution set" for the two-candidate game with determininstic voters. However since it appears that the uncovered set lies inside the heart, chaotic or cyclic behavior by candidates is possible.

In the probabilistic model on the contrary, the assumptions on voter behavior (Enelow and Hinich 1984; Coughlin 1992) are sufficient to ensure that the win probabilities are continuous in the candidate strategies. Indeed in these models it is usual to use, not the win probabilities (ξ_1, ξ_2) but the expected votes $\{\pi_1(z_1, z_2), \pi_2(z_1, z_2)\}$. Under conditions of voter independence these two models give the same Nash equilibria (see Aranson, *et al.* 1974 and the chapter by Ladha in this volume). However the work by Ladha and Miller (this volume) can be interpreted to imply that the probability of winning and expected vote do not give equivalent candidate objective functions when voter choices are statistically dependent. To obtain $PSNE$ it is usually assumed that the *voter* probability functions have a convenient form (see the discussion in Chapter 1 of this volume), so as to guarantee that the best response functions $\{R_1, R_2\}$ are quasi-concave.

In a sense then the analysis of two party models of this kind come down to an examination of the equilibrium correspondence

$$E_g \; : \; U(X_1 \; \times \; X_2) \; \times \; U(X_1 \; \times \; X_2) \; \longrightarrow \; 2(\tilde{X}_1 \; \times \; \tilde{X}_2).$$

Here, the notation $U(X_1 \times X_2)$ is meant to represent the class of utility functions defined on both z_1 and z_2. That is $E_g(U_1, U_2)$ is obtained by finding the fixed points of the best response correspondence $R(U_1, U_2)$, thus giving the set of $MSNE$ for the game form g. Typically, of course, these models are posed in the context of a distribution, f, of voters and the equilibrium outcome is one where both parties choose the same point.

While the above model gives an elegant description of electoral politics, it is not evident that two party competition does, in fact, produce "Downsian convergence" as suggested by the probabilistic model.

There are many empirical situations of two parties competing in a way that suggests that they both wish to win, yet with little indication of convergence in declarations. If the parties are indeed "Downsian", then the probabilistic model in such a case must be invalid. The model could be modified by considering different voter probability functions $\{\pi_{ij}, \pi_{ik}\}$ or by assuming voter dependency. A different possibility is to suppose that parties' utility functions depend on the policy outcomes. Cox (1984a) for example, assumes that each party i has a bliss point x_i in X where its "political preferences" are maximised. Normally it is assumed that $u_i(z) = -\frac{1}{2} \| z - x_i \|^2$.

The expected utility model can now be written

$$\begin{aligned} U_1(g(z_1, z_2)) &= \xi_1(z_1, z_2)(u_1(z_1) + \gamma_1) + \xi_2(z_1, z_2)u_1(z_2) + \\ &\quad (1 - \xi_1 - \xi_2)(u_1(\tfrac{(z_1+z_2)}{2}) + \tfrac{\gamma_1}{2}). \qquad (Eq.2) \end{aligned}$$

and similarly for party 2.

In this model party 1 presents a "manifesto" z_1 to the electorate, promising that this policy position will be played should i win the election. The prize γ_1 is the "perquisite" of government. In the event of a draw, we can assume (as we have in Eq. 2) that the parties agree to implement the policy $\frac{z_1+z_2}{2}$ and to split the perquisite.

In this case the equilibrium correspondence can be written:

$$E_g : U(X) \times U(X) \longrightarrow 2(\tilde{X}_1 \times \tilde{X}_2)$$

since the fundamental parameters concern the policy preferences, and these are described by utility functions with domain X. (Of course, nothing is changed by linear transformations of utilities, so we should regard $U(X)$ as the space of von Neumann-Morgenstern "preferences" over finite lotteries on X.) In the usual applications however the preferences are assumed to be Euclidean. Without great loss of generality, therefore, we may write

$$E_g : X_1 \times X_2 \longrightarrow 2(\tilde{X}_1 \times \tilde{X}_2)$$

for the equilibrium map.

Note that this model has two features. It is "policy-seeking" in the sense that the utilities u_i are dependent on policy bliss points $\{x_i\}$ and the policy outcome. It is "office-seeking" since each party gains an award, or perquisite, γ_i from winning. We can contrast the pure policy-seeking model (when each $\gamma_i = 0$), with the pure office-seeking or "Downsian" model when $u_i(z_i) = 0$. The model is set up with electoral risk, but we could instead define it in terms of uncertainty–namely probability distributions over win probabilities.

There is one serious difficulty with all policy-seeking models such as this. Having won the election with declaration z_i, why would party i implement policy z_i rather than its preferred policy x_i? This suggests that in the full equilibrium analysis each party should compare implementing z_i against the later electoral costs of implementing x_i instead. From the point of view of the electorate, why should they believe candidate i will implement its declaration z_i if it wins? A number of interesting papers (Austen-Smith and Banks 1989; Banks 1990; Banks and Sundaram 1990, 1993) have tackled this problem. Notwithstanding these attempts, I believe this problem of credible commitment by parties is not yet resolved. On the contrary, I believe a case can be made that the uncertainty involved in the degree of credible commitment should be incorporated into the win probabilities ξ_i, ξ_j. This becomes very complex however, because there would need to be a strong component of common knowledge in the beliefs by parties about the electorate, and in the beliefs by the electorate about the parties. This we might describe as *political culture*.

Attempts to develop this model in the multiparty context are suggestive, and the model I propose can be regarded as a natural generalization. Suppose that the set $N = \{1, \ldots, n\}$ of parties transmits a *message profile* $z = (z_1, \ldots, z_n)$ to the electorate, where z_i is intended to represent the information that party i chooses to transmit to the electorate about its intentions. For simplicity I shall assume that the message from party i is simply an "ideal point", z_i, with the implicit assumption that i will act after the election as though it had Euclidean policy preferences based on z_i. However, I shall also assume that the "true" party preferences (based on a "bliss" point, x_i) are unknown to the voters. Electoral preferences are described by a distribution, f of voter "bliss" points.

Under deterministic voter behavior, the electoral response is described by an electoral map $e^f : X^N \longrightarrow \Delta_N$ where $e^f(z) = (e_1(z), \ldots, e_n(z))$ gives the vector of seat shares to each party. Since $\sum_{i=1}^{n} e_i(z) = 1$, the image of the electoral map is in the $(n-1)$ dimensional simplex Δ_N.

David Austen-Smith argues (in Chapter 6 of this volume) that a fully developed model of voter behavior should analyze voter strategies as functions not only of the declarations, z, but of the legislative outcome function. For the present I shall assume that e^f results from "sincere" voter strategies.

Eaton and Lipsey (1975) considered a version of the "Downsian" model where each party has a utility function $u_i(z) = e_i(z)$. That is each party attempts to maximize the number of seats it controls. In general the best response correspondence $R(u_1, \ldots, u_n) : X^N \longrightarrow 2(X^N)$ has no fixed point (Shaked 1975) when the dimension is 2 and $n = 3$. Consequently there may be no $PSNE$. Many authors have attempted to show existence of equilibrium where the parties in parliament attempt to keep out intruders (Greenberg and Weber 1985; Greenberg and Shesple 1987; Palfrey 1984; Shepsle 1991; see also Chapter 6 of this volume for a more detailed discussion.)

The general difficulties with existence of equilibrium in this model and its variants are two-fold: (i) discontinuity; and (ii) non-convexity in induced preference. A discontinuity can occur at a message profile $z = (z_1, \ldots, z_n)$ where $z_i = z_j$ for all i, j. In this case it is reasonable to assume $e_i(z) = e_j(z) = \frac{1}{n}$, for all i, j. Clearly a small perturbation to $z' = (z'_1, \ldots, z'_n)$ can result in discontinuous change in $e_i(z')$. Dasgupta and Maskin (1986) have argued that this is not a fundamental problem. They propose a *standard smoothing* procedure. That is consider a profile $z' = (z'_1, \ldots, z'_n)$ with $z'_i = z'_j$. Then it is possible to smooth away the discontinuities by defining a new continuous outcome function $\overline{e}$ with $\overline{e}(z) = e(z)$ for any $z \notin V_\epsilon(z')$, an ϵ−neighborhood of z', but also satisfying $\overline{e}(z') = e(z')$. Moreover the pure strategy (or mixed strategy) Nash equilibria determined by e and $\overline{e}$ coincide. Dasgupta and Maskin (1986) showed that mixed strategy Nash equilibria exist in a general class of such models.

Even assuming the relevance of the model, what precisely is the interpretation of a mixed strategy Nash equilibrium? The more general question however is: Why would parties in a multiparty context attempt to maximize seats?

It is difficult to believe that parties have no policy preferences, and no ideological positions to maintain. If they do have policy preferences, then there is a cost to them of presenting a policy proposal to the electorate simply to gain seats. Of course they need seats to obtain representation in the Parliament, so there is an implicit trade-off between policy choice and electoral advantage, which the pure multiparty office-seeking or Downsian models do not address. There is also a serious difficulty in using a variant of Cox's (1984a) mixed policy-seeking office-seeking model. In a multiparty parliamentary system, it is very uncommon for any party to gain a straight majority of the seats. Data presented in Schofield (1993b) indicate that only 17 majority party situations occurred out of 216 governments in twelve European countries examined in the period from 1945 until 1987. To determine whether a Nash equilibrium exists in the selection of party manifestos we must model a dependence of u_i on z that is not simply based on the number of seats won by the party.

One modelling strategy, adopted by Baron (1989, 1991), is to assume that each party declares its "true" bliss point. He considers a model where there are three parties. Each party, i, has true Euclidean preference with bliss point x_i, which it declares. The outcome is a finite lottery

$$g(x) = \{y_1, \rho_1(x);\ y_2, \rho_2(x);\ y_3, \rho_3(x)\}.$$

Here $x = (x_1, x_2, x_3)$ are the three bliss points and $y_i(x)$ is an outcome associated with coalition $\{j, k\}$ that occurs with a probability $\rho_i(x)$ that is structurally determined by x and by exogeneous parameters.

The problem with this model is that it may not be incentive compatible. In particular since g is defined for any manifesto profile $z \in X^N$ and U_i can be defined as the von Neumann-Morgenstern expected utility function obtained from the underlying Euclidean preferences, it is likely, for party 1, say, that there exists $z_1 \in X$ such that

$$U_1(z_1, x_2, x_3) > U_2(x_1, x_2, x_3).$$

In a later paper Baron (1993) considered a variant of this model, where the parties are "endogenous". He assumed that parties are collections of voters, who choose declarations so as to maximize average electoral utility (via $g(z)$) within the group. This is an ambitious model, but I believe it misses the essential feature of principal agent relations in representative politics.

An influential paper by Austen-Smith and Banks (1988) dealt with the problem in an ingenious fashion. The space is one-dimensional and there are three parties. Once the profile of declarations $z = (z_1, z_2, z_3)$ is selected, then given the electoral map $e^f : X^3 \longrightarrow \Delta_3$ there is a unique outcome $g(z)$ specifying the governing coalition M, a policy point z_M and perquisites $\{\gamma_i : i \in M\}$. Knowing g, the electorate chooses in an equilibrium fashion. (See Chapter 6 of this volume for a full discussion.) Thus both the electorate and the parliamentary parties make choices that are in equilibrium with each other. However, though the preferences of the electorate are exogenous, the party preferences are endogenous. In particular, the outcome $g(z)$ is evaluated by party i (if it enters the government coalition M) using the endogenous utility $u_i = - \| z_i - z_M \| + \gamma_i$. That is, the declaration, z_i, chosen by i is used as the "induced" bliss point by that party. The problem of credible commitment by i to z_i does not arise.

In the next section of the chapter, I construct a model that is based on parties' "exogenous" policy preferences and focuses on the existence of a Nash equilibrium in coalition bargaining.

8.5 Coalition Bargaining in a Committee

This section attempts to construct a multiparty model of coalition bargaining that is a natural generalization of the work, described in Section 8.4, of Austen-Smith and Banks, Baron and Cox.

We consider first of all the case of three parties $N = \{1, 2, 3\}$ and X a compact, convex subset of $\Re^2$. Once an adequate model is obtained in this case then extending it to large n and more general X should, in principle, be easy. We also ignore the electoral connection (for the moment). That is $\mathcal{D}$ is fixed and equal to $\{\{1,2\},\{2,3\},\{1,3\}\}$. Thus the framework is based on a model of spatial voting in a committee. However it can be viewed as resembling Austria or Germany where any two party coalition can form a government. Because $w(\mathcal{D}) = 2$ as we have seen, the core (if it ever exists) is unstable. Each party, i, has "true policy preferences" defined by a "policy" utility function $u_i(z) = -\frac{1}{2} \| z - x_i \|^2$, where z is an outcome in X. Note that each bliss point, x_i, is chosen by a process not considered by the model. If party i forms a government with j, then i receives a perquisite γ_{ij}, and the political utility function is separable in policy and perquisites.

We may therefore regard the model as an adaptation of spatial voting, but incorporating certain fixed benefits $\{\gamma_{ij}\}$ that are allocated to members of the final winning coalitions. The innovation of this model is that, instead of supposing that some geometrically defined set of outcomes in X is possible from coalition bargaining, it proposes that each party makes an offer (a "declaration" of proposed policy). The "legislative outcome" resulting from a vector $z = (z_1, z_2, z_3)$ of declarations is a finite lottery $g(z)$ across coalition-policy outcomes and distributions of perquisites. This defines a game, which may or may not have a *PSNE*. Unlike the models of electoral competition, involving electoral risk, this model involves coalitional risk. That is, each two party coalition, M, forms with probability, ρ_M, where ρ_M is inversely related to the "distance" between the declarations of the coalition members. Some empirical evidence for such an assumption is implicit in the discussion below in Chapters 12 and 13, of the "proto-coalition model".

Assumptions of the 3-Party Model.

1. Once party i chooses a declaration $z_i \in X$ it acts in negotiation with other parties as though it had a Euclidean preference given by $u_i'(z) = -\frac{1}{2} \| z - z_i \|^2$. That is, it is committed to z_i in coalition bargaining. The declaration, z_i, may be viewed as an offer to the other parties.

2. Once the profile $z = \{z_1, z_2, z_3\}$ is declared, then the policy outcome is a lottery $g(z) = \{(\rho_M, z_M) : M \in \mathcal{D}\}$ where $z_M = \frac{1}{|M|} \sum_{i \in M} z_i$ and $\rho_M(z)$ is the probability that coalition M forms.

3. For each $M = \{i, j\}$ if $z_i \neq z_j$ then $\rho_M(z)$ is inversely proportional to $\| z_i - z_j \|^2$. For each z, the sum $\sum \{\rho_M(z) : M \in \mathcal{D}\} = 1$.

4. The three functions $\rho_M : X^3 \longrightarrow [0, 1]$ are smoothed near any point z such that $z_i = z_j$.

5. Each profile choice $z \in X^3$ determines an outcome $g_\gamma(z) \in \tilde{X}_\Delta$ where $\tilde{X}_\Delta$ is the space of finite lotteries over policy outcomes and perquisites. That is, $g_\gamma(z) = \{(\rho_M, z_M, \gamma_M)\}$ where γ_M is a specified vector of perquisites to members of M. It is assumed here that the perquisites to the two members $\{i, j\}$ of M are exogenously specified. This assumption could easily be modified. We also write $g(z) = \{(\rho_M, z_M)\}$ for the *policy lottery outcome* in $\tilde{X}$. Each party, i, evaluates the outcome $g_\gamma(z)$ by a von Neumann-Morgenstern utility function $U_i : \tilde{X}_\Delta \longrightarrow \Re$, which is separable in policy preferences and perquisites and based on the true Euclidean policy preferences.

6. Each party, i, chooses a best response to the choices z_j, z_k of the parties by maximizing U_i. That is, i's best response is given by a correspondence $R_i : X_j \times X_k \longrightarrow X_i$ where X_j and X_k are the strategy spaces of j, k (namely copies of X).

7. The pure strategy Nash equilibrium (if one exists) is a fixed point of the joint response correspondence

$$R_{123} : X^3 \longrightarrow 2(X^3).$$

As before the $MSNE$ belongs to $2(\tilde{X}^3)$. The smoothing operation of (4) is carried out to avoid the discontinuities observed by Dasgupta and Maskin (1986) near the diagonal. This problem is easy to avoid by constraining $\rho_{ij}(z) \leq 1 - \epsilon$, for some $\epsilon > 0$ in a neighborhood of any profile z with $z_i = z_j$, and by smoothing the ρ_{ij} in a neighborhood of a profile with $z_1 = z_2 = z_3$. (See Schofield and Parks, 1993, for details.)

Notice in this model that we have assumed a specific game form $g_\gamma : X^3 \longrightarrow \tilde{X}_\Delta$. This one seems very natural to analyze coalition bargaining. Based on the declarations of intended policies, z_i, z_j by two parties, they choose a compromise position to implement in case they form a government. Moreover in the *symmetric* case when $\| z_1 - z_2 \| = \| z_2 - z_3 \| = \| z_1 - z_3 \|$, then each of the three two-party coalitions are equally likely, so that $\rho_{ij} = \frac{1}{3}$.

To analyze this game we need to determine the properties of the equilibrium map

$$E_g : X^3 \longrightarrow 2(\tilde{X}^3),$$

which takes a profile of bliss points (x_1, x_2, x_3) to the $MSNE$.

Let $\gamma = \{\gamma_{ij} : i, j = 1, 2, 3\}$ be the pattern of party perquisites in the various government coalitions. For each Euclidean profile, represented by the profile of bliss points $x = (x_1, x_2, x_3) \in X^3$, does there exist a fixed point $E_g(x)$ to the joint response correspondence? If there is no $PSNE$, then there may be an $MSNE$, $E_g(x) \in \tilde{X}^3$.

Schofield and Parks (1993) have recently shown analytically that a $PSNE$ exists for particular profiles in X^3, when the game form has certain features.

Definition 1:

1. Say three points $\{z_i, z_j, z_k\}$ each in $\Re^m$ are $\epsilon-$ *bounded in linearity* if

$$min_{\lambda_j, \lambda_k \in \Re} \{\| z_i - \lambda_j z_j - \lambda_k z_k \|\} \leq \epsilon.$$

If $\epsilon = 0$ the points are *colinear.*

2. Say three points $\{z_i, z_j, z_k\}$ are $\epsilon-$ *bounded in symmetry* if

$$max_{i,j,k} \mid \| z_i - z_k \| - \| z_j - z_k \| \mid \leq \epsilon$$

where $max_{i,j,k}$ means across all permutations of i, j, k. If $\epsilon = 0$ then the three points are *symmetric.*

3. Say three points $\{z_1, z_2, z_3\}$ are $\epsilon-$ *convergent* if there exists a ball, B, of radius ϵ, and center in the convex hull of $\{z_1, z_2, z_3\}$ which contains all three.

Theorem 1: *In the three party model of coalition formation on* $\Re^2$, *with zero perquisites, then there exists* $\epsilon > 0$ *such that*

1. *when the bliss points $x = (x_1, x_2, x_3)$ are ϵ−bounded in linearity there exists a unique PSNE, $E_g(x)$. Moreover $E_g(x) = (z_1, z_2, z_3)$ such that these points $\{z_1, z_2, z_3\}$ are ϵ'-convergent, for some ϵ' dependent on ϵ,*

2. *when the bliss points are ϵ−bounded in symmetry, there exists a unique PSNE, $E_g(x)$. Moreover $E_g(x) = (z_1, z_2, z_3)$ such that $\{z_1, z_2, z_3\}$ are ϵ'-bounded in symmetry for some ϵ' dependent on ϵ. In particular, if $\epsilon = 0$, then $\{z_1, z_2, z_3\}$ are symmetric and $\| z_i - z_j \| = 2 \| x_i - x_j \|$ for each pair i, j.*

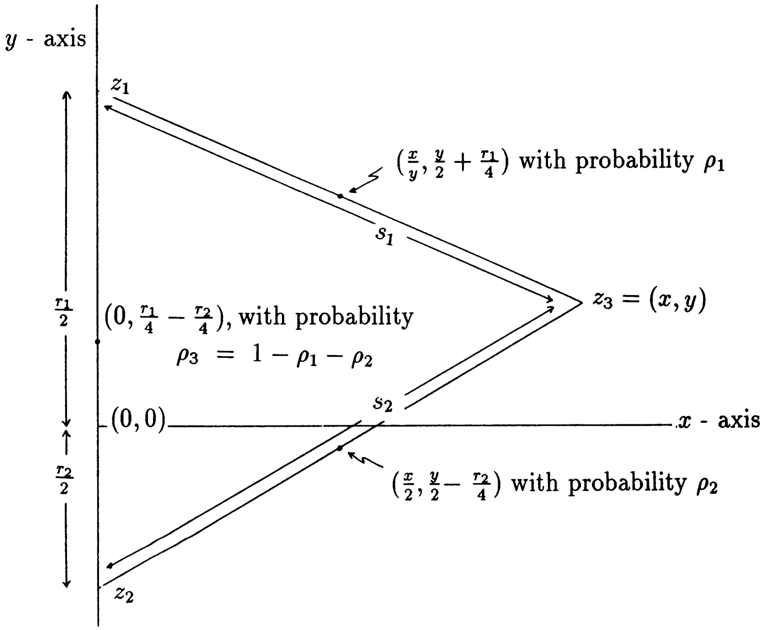

Figure 5. Lottery of three possible legislative outcomes, given three party manifesto positions (z_1, z_2, z_3).

To illustrate this result, consider Figure 5 which presents an example where three parties have adopted positions (z_1, z_2, z_3) and coordinates are chosen so that $z_1 = (0, \frac{r_1}{2}), z_2 = (0, -\frac{r_2}{2}), z_3 = (x, y)$. (It is hoped that coordinate x is not confused with the bliss point x_i.) For convenience suppose that the bliss point for party 3 is at $(L, 0)$. Now consider the best response problem from the point of view of party 3. To simplify notation, let us write ρ_1 for ρ_{31}, ρ_2 for ρ_{32} and $1 - \rho_1 - \rho_2$ for ρ_{12}. Remember γ_{3i} is the perquisite to party 3 if coalition $\{3, i\}$ forms. From the point of view of party 3, the outcome $g_\gamma(z)$, resulting from the joint declaration, z, consists of three possibilities:

1. policy $\frac{z_1 + z_3}{2} = (\frac{x}{2}, \frac{y}{2} + \frac{r_1}{4})$ and perquisite γ_{31}, with probability ρ_1

2. policy $\frac{z_2 + z_3}{2} = (\frac{x}{2}, \frac{y}{2} - \frac{r_1}{4})$ and perquisite γ_{32}, with probability ρ_2, and

3. policy $\frac{z_1+z_2}{2} = (0, \frac{r_1-r_2}{4})$ and no perquisite with probability $1-\rho_1-\rho_2$.

Notice that $\| z_1 - z_2 \| = \frac{1}{2}(r_1 + r_2)$. Let us write $s_i = \| z_3 - z_i \|$ for $i = 1, 2$. Thus

$$\rho_i = \left(\frac{1}{s_i^2} + \frac{1}{s_j^2} + \left(\frac{2}{r_1 + r_2}\right)^2\right)^{-1} \frac{1}{s_i^2}.$$

Clearly ρ_1, ρ_2 can be effected by choice of x and y. If we write $U(x,y)$ for the von Neumann Morgenstern expected utility for 3 in terms of the choice variables $(x,y) \in \Re^2$, and u for this party's true Euclidean utility, then the response problem for 3 is to maximise

$$U(x,y) = \rho_1 \left\{u\left(\tfrac{z_1+z_3}{2}\right) + \gamma_{31}\right\} + \rho_2\left\{u\left(\tfrac{z_2+z_3}{2}\right) + \gamma_{32}\right\} + (1-\rho_1-\rho_2)\left\{u\left(\tfrac{z_1+z_2}{2}\right)\right\}.$$

Notice the similarity of this equation to *Eq.* 2 governing two party competition (Cox 1984a).

Parks and Schofield (1993) have calculated "best response" by taking partial derivatives. These give "optimality" equations in x and y. Taking $\frac{\partial U}{\partial x} = 0$ gives

$$\frac{\rho_1 + \rho_2}{2}\left(L - \frac{x}{2}\right) - 2x\left(\frac{\rho_1^2 + \rho_2^2}{\overline{r}^2}\right)\left(\frac{\gamma_{31} + \gamma_{32}}{2} + Lx - \frac{x^2}{4} - \epsilon\right) = 0 \quad (Eq.3)$$

where $\overline{r} = \frac{r_1+r_2}{2}$ and $\epsilon = \frac{r^2}{16}$.

Equation $\frac{\partial U}{\partial y} = 0$ gives $y(\rho_2 + \rho_2) =$

$$-\frac{1}{2}(\rho_1 r_1 - \rho_2 r_2) - 8y\left(\frac{\rho_1^2}{\overline{r}^2} + \frac{\rho_2^2}{\overline{r}^2}\right)\delta + 4\left(\frac{r_1\rho_1^2}{\overline{r}^2} - \frac{r_2\rho_2^2}{\overline{r}^2}\right)\delta, \quad (Eq.4)$$

where $\delta \simeq \frac{\gamma_{31}+\gamma_{32}}{2} + u(\frac{x}{2}, \frac{y}{2}) - u(0, \frac{r_1-r_2}{4})$.

These rather difficult non-linear equations can be easily solved in the case $r_1 = r_2$ $(= r)$. Then $\rho_1 = \rho_2$ and by symmetry the optimal solution for y is $y^* = 0$. If $\gamma_{31} = \gamma_{32} = 0$, then the equation in x becomes

$$\rho\left\{\left(L - \frac{x}{2}\right) - \frac{2x\rho^2}{r^2}\left(Lx - \frac{x^2}{4} - \frac{r^2}{16}\right)\right\} = 0. \quad (Eq.5)$$

To solve this equation, consider the case $L = \beta r, x = ar$. We obtain

$$\rho\left\{\left(\beta - \frac{a}{2}\right) - 2a\rho\left(a\beta - \frac{a^2}{4} - \frac{1}{16}\right)\right\} = 0. \quad (Eq.6)$$

Now $\rho = \frac{r^2}{2r^2+s^2} = \frac{1}{\frac{9}{4}+a^2}$ since $s^2 = (\frac{r}{2})^2 + x^2$. Assuming $\rho \neq 0$, gives a quadratic expression in a, β with solution

$$a = \frac{1}{2}\left(-\frac{1}{\beta} \pm \sqrt{\frac{1}{\beta^2} + 9}\right). \qquad (Eq.7)$$

Thus if $L \gg r$, so that $\beta \gg 1$, then $x^* = \frac{3r}{2}$. Notice that there is only one positive solution, so that $U(x,0)$ is concave on the positive $x-$axis.

Note however that if $x = \frac{3r}{2}$, then $s^2 = (\frac{r}{4})^2 + x^2 > r^2$. Thus best response produces a divergence of the position of party 3 from the positions of parties 1 and 2. To examine the symmetric case further, suppose that $\beta \simeq 1$. It is easy to show that if $\beta = \frac{\sqrt{5}}{2}$, then $a = \beta$. In particular, if $\beta \in (\frac{\sqrt{5}}{2}, \infty)$, then $a \in (\frac{\sqrt{5}}{2}, \frac{3}{2})$ with $a < \beta$, while if $\beta \in (0, \frac{\sqrt{5}}{2})$, then $a > \beta$, but $a \in (0, \frac{\sqrt{5}}{2})$. If $\beta = \frac{1}{\sqrt{3}}$, then $a = \frac{\sqrt{3}}{2}$, and $s^2 = r^2$.

Given z_1, z_2 there is a 1-1 relationship between the bliss point $(\beta r, 0)$ and the best response $(ar, 0)$. In particular if $L = \frac{r}{\sqrt{3}}$, then $x^* = \frac{\sqrt{3}r}{2}$ and the best response is to set $s = r$.

Consequently, if the bliss points are symmetric then the joint best response will also be symmetric. Further analysis shows that there exists a pure strategy Nash equilibrium $\{z_1, z_2, z_3\}$ related to the bliss points $\{x_1, x_2, x_3\}$ by $\| z_i - z_j \| = 2 \| x_i - x_j \|$.

For bliss points close to symmetry, transversality arguments show that the *PSNE* is again unique and also close to symmetric.

Suppose now that the declarations are $z_i = (0, y_i)$ with $y_1 < y_2 < y_3$. For such colinear points it is possible to compute $\frac{\partial U}{\partial y}$. Then $\frac{\partial U}{\partial y}$ is maximized at $x^* = 0$, with y^* equal to the midpoint of $\{y_1, y_2, y_3\}$. Simulation of this optimization problem when the bliss points are close to colinear shows that the unique Nash equilibrium occurs when all three parties choose positions in some $\epsilon'-$neighborhood of the middle bliss point.

The conclusion from this analysis is somewhat surprising. For bliss points near to symmetric, divergence in declared positions occurs. However if the bliss points are close to colinear, then a form of Downsian convergence occurs.

It is also possible to determine the effects on the Nash equilibrium when government perquisites are added.

Theorem 2: *In the 3-party model of government formation, if government perquisites $\{\gamma_{ij}\}$ are of the same order of magnitude as policy disagreements $(\| x_i - x_j \|^2)$ then there exists a unique $\epsilon-$convergent pure strategy Nash equilibrium, where ϵ is dependent on $\{\gamma_{ij}\}$.*

To illustrate this consider Eq. 3 with an additional perquisite term given by $\gamma_{31} = \gamma_{32} = r^2$. Then Equation 7 becomes

$$a = \frac{1}{2}\left(-\frac{3}{\beta} \pm 3\sqrt{\frac{1}{\beta^2} + 1}\right).$$

If $\beta = \sqrt{3}$ then $a = \frac{\sqrt{3}}{2}$. Consequently if $\gamma_{ij} \simeq r^2$, all i, j, then the best response correspondence is a contraction mapping which has a unique

Nash equilibrium. The greater the magnitude of the perquisites, the greater the degree of convergence (*i.e.*, the smaller is ϵ).

It is conjectured that these two results hold for a general form of this model involving n parties, where the probabilities ρ_M are inversely proportional to the variance of $\{z_i : i \in M\}$. It is also possible that such a model could be used to explain the formation of parties. In simulation of this game, Schofield and Parks (1993) found, in the case of zero perquisites, that parties "close" to one another tend to converge, while parties far from the others diverge. Thus, in Nash equilibrium, there is a complex pattern of convergence and differentiation.

On the other hand, if perquisites dominate, as in the office seeking "Downsian" model, convergence always occurs.

We can think of these results in more general terms. The construction of the game form g_γ gives, for each z, a finite lottery $g_\gamma(z)$ in the space $\tilde{X}_\Delta$. Moreover, since $\mathcal{D}$ is fixed, the heart $\mathcal{H}_\mathcal{D}(z)$ is the triangle with vertices $\{z_1, z_2, z_3\}$. Each possible policy outcome under the game form belongs to $\mathcal{H}_\mathcal{D}(z)$, so the *policy lottery outcome*, namely $g(z) \in \tilde{X}$, belongs to $\tilde{\mathcal{H}}_\mathcal{D}(z)$, the space of finite lotteries over the set $\mathcal{H}_\mathcal{D}(z)$. As z varies, then $g(z)$ varies continuously with $\tilde{\mathcal{H}}_\mathcal{D}(z)$. Since $\mathcal{H}_\mathcal{D}$ is lower hemi-continuous, so is $\tilde{\mathcal{H}}_\mathcal{D}$ (Schofield 1993a). Thus we see that $g : X^3 \longrightarrow \tilde{X}$ is a *continuous selection* of the correspondence $\tilde{\mathcal{H}}_\mathcal{D} : X^3 \longrightarrow 2(\tilde{X})$. Notice also that the expectation of g at the Nash equilibrium outcome always belongs to the Pareto set (the convex hull of $\{x_1, x_2, x_3\}$).

Combined analysis and simulation give the following:

Theorem 3: *In the 3-party model:*

1. *If perquisites are identically zero, then the expectation of the policy outcomes, in Nash equilibrium, namely* $\underline{Ex}[g(E_g(x))]$ *belongs to the Pareto set of the profile* $x = \{x_1, x_2, x_3\}$.

2. *If all perquisites exceed some level* $\overline{\gamma}$, *determined by the bliss points* $\{x_1, x_2, x_3\}$ *then all PSNE outcomes in* $g(E_g(x))$ *belong to the Pareto set of the profile. (Since* $\mathcal{D}$ *is fixed, the Pareto set is simply* $\mathcal{H}_\mathcal{D}(x)$.*)*

To see how to extend this model to one where there is dependence on electoral response to n parties, let $\mathcal{D}(z)$ be the electorally-determined set of decisive coalitions at the profile, z, of declarations. Let $g(z)$ be the lottery of coalitionally determined policy outcomes. It is natural to assume that $g : X^N \longrightarrow \tilde{X}$ is a continuous selection of the heart correspondence $\tilde{\mathcal{HD}} : X^N \longrightarrow 2(\tilde{X})$. Here $\tilde{\mathcal{HD}}(z)$ is the set of finite lotteries across the heart, as defined by z and $\mathcal{D}(z)$. The previous result that $\mathcal{H}_\mathcal{D}$ is *lhc* can be extended to show the correspondence $\tilde{\mathcal{H}}_\mathcal{D}$ is *lhc* (Schofield 1993a) and so we do know that there does exist such a selection, g.

For example, if the declaration profile z gives rise to a distribution of seats such that one party (say SD) is the core, then $g(z) = \{z_{SD}\}$ the declaration of that party. If in fact a minority SD government forms, then the vector γ of perquisites would assign all the cabinet positions to that party.

The next section suggests how to extend this model to deal with electoral risk.

8.6 The Heart and Electoral Politics

In Section 8.3 we argued that the heart gave a theoretical way to interpret coalition bargaining in multiparty systems such as those of a number of European countries. However there was an apparent problem to resolve. The heart was defined earlier in terms of true preferences (and bliss points), whereas the empirical evidence was obtained from factor analytical methods applied to party manifestos or policy declarations. In Section 8.5 we considered a three-party competitive model involving no electoral competition. Instead parties chose declarations to which they were committed in policy bargaining. We assumed that these declarations were chosen in Nash equilibrium under a common knowledge assumption about the game form, g_γ, mapping from declarations to the lottery of outcomes, in $\tilde{X}_\Delta$. Reasons were given why the Nash equilibrium correspondence

$$E_g : X^N \longrightarrow 2(X^N)$$

from Euclidean profiles to equilibrium outcomes would be single-valued for general N and "reasonable" g.

The heart model of Section 8.3 describes post-election party bargaining when seat strengths, represented by $\mathcal{D}$, are known. On the other hand the competitive model of Section 8.5 describes pre-election behavior when $\mathcal{D}$ is assumed fixed. Note that we showed in Section 8.5 that the expected value of the equilibrium outcome always belongs to the Pareto set of the true bliss points (x_1, x_2, x_3). Moreover the policy components of the equilibrium, $z^* = E_g(x)$, always belongs to the "lottery" heart of the declarations $\tilde{\mathcal{H}}\mathcal{D}(z^*)$.

To develop a model of $n-$party competition we proceed as folllows. Once the vector of policy declarations $z = (z_1, \ldots, z_n)$ is known then each voter, v, responds by a probability vector $\pi_v(z) = (\pi_{v1}(z), \ldots, \pi_{vn}(z))$, where $\pi_{vi}(z)$ is the probability that v chooses party i. Let $(\overline{\pi}_1, \ldots, \overline{\pi}_n)$ be the vector of expected votes for each party. Of course the vote for each party is now a random variable defined by a probability distribution. With at least three parties it is no longer possible to say whether one party or the other wins. However if we let $\{\mathcal{D}_t\}$ be the class of possible winning coalition structures, then we can construct a probability distribution over $\{\mathcal{D}_t\}$. Specifically, let $\xi_t(z)$ be the probability that the winning coalition structure, $\mathcal{D}_t$, results from the profile of declarations, z, given the electoral response $\{\pi_v(z)\}$. Let $\{\xi_t(z)\}$ be this list of probabilities defined at z. For example in the case of two parties $\{i,j\}$, $\{\mathcal{D}_t\} = \{\{i\},\{j\},\{i,j\}\}$ where $\{i,j\}$ means that i and j draw.

In the case of three parties $\{i,j,k\}$ clearly $\{\mathcal{D}_t\} = \{\{i\},\{j\},\{k\},\{i,j\},\{j,k\},\{i,k\}\}$ since either one or two parties must win under majority rule. It is possible, of course, to have more complex winning coalition structures, but this formulation obviously generalizes the simple two-party case.

Note however that, in the two party case, maximizing $\overline{\pi}_i$ and maximizing the probability of winning are equivalent. In the multiparty case it appears necessary to consider the probabilities rather than expected vote. To pursue the existence of equilibria, I shall assume that ξ_t are continuous functions of the vector, z, of declarations. This implicitly makes certain assumptions on voter behavior, which we leave unspecified in the present work (see however Nixon, *et al.,* 1995, and David Austen-Smith's discussion in Chapter 6).

Given a coalition structure, $\mathcal{D}_t$ at the policy vector z, let $g_{\gamma(t)}(z)$ be the "legislative" outcome. That is, $g_{\gamma(t)}(z)$ defines a specific policy lottery, $g_t(z)$, giving probabilities $\{\rho_M\}$ and outcomes $\{z_M\}$ associated with each coalition $M \in \mathcal{D}_t$. Moreover $g_{\gamma(t)}(z)$ defines for each M an allocation γ_M of perquisites to each member of M. Now let $\tilde{X}_\Delta$ be the set of finite lotteries over the policy space X and the simplex, Δ, of distributions of perquisites.

We may put electoral risk and coalition risk together in the form of a general model of $n-$party competition.

Assumptions of the General n-Party Model.

1. The electoral response ξ to a joint policy declaration $z = (z_1, \dots, z_n)$ is an assignment $\{\xi_t(z) : t = 1, \dots, T\}$ of probabilities, one to each possible decisive structure, $\mathcal{D}_t$. Let Δ_T be the $(T-1)$ dimensional unit simplex.

   Then the map $\xi \; : \; X^N \longrightarrow \Delta_T : z \longrightarrow (\xi_1(z), \dots, \xi_T(x))$ is continuous and common knowledge.

2. The outcome function is a continuous map $g_\xi : X^N \longrightarrow \tilde{X}_\Delta$ defined from ξ by $g_\xi(z) = \{(\xi_t(z),\, g_{\gamma(t)}(z)) \; : \; t = 1, \dots, T\}$.

Here $g_{\gamma(t)}(z)$ specifies the legislative outcome given the declarations of the parties and given the decisive structure $\mathcal{D}_t$.

We shall also assume, as we did in Section 8.5, that $g_{\gamma(t)}(z)$ is "separable" between policy outcomes and perquisites, and that the policy component $g_t \; : \; X^N \longrightarrow \tilde{X}$ is a continuous selection from the heart correspondence, $\mathcal{HD}_t \; : \; X^N \longrightarrow 2(\tilde{X})$. That is, if coalition structure $\mathcal{D}_t$ occurs after the election, then $g_{\gamma(t)}(z)$ selects certain coalition outcomes from $\mathcal{HD}_t(z)$. In this way we deal both with electoral risk (represented by ξ) and political or coalitional risk (ρ represents the indeterminacy underlying coalition formation). Because of the assumption of von Neumann-Morgenstern utility functions, the lottery outcome $g_\xi(z)$ can be evaluated by each party.

In principle it is possible to impose quite a general form on the coalition policy outcomes. The structure of the model attempts to generalize both the pure two party model of electoral risk proposed by Cox (1984a) and the pure three-party model of coalitional risk proposed by Baron (1989).

A Nash equilibrium of the electoral game at the profile $x = (x_1, \dots, x_n)$ is just an equilibrium of the game form g_ξ. In general a $PSNE$ may not exist, but because of the assumptions on the continuity of each ξ_t and g_ξ there will exist an $MSNE$, described by an equilibrium map $E_{g_\xi} \; : \; X^N \longrightarrow \tilde{X}^N$. If we extend g_ξ to the domain $\tilde{X}^N$, then the equilibrium outcome is the lottery $g_\xi(E_{g\xi}(x)) \in \tilde{X}_\Delta$. This in turn defines an expectation $\underline{Ex}[g(E_{g\xi}(x)]$ for the policy outcome.

The model is rather complex but the idea is less difficult. Each party (1, say) may make reasonable guesses as to the declarations $\{z_2, \ldots, z_n\} = z_{-1}$ of the other parties. For each possible choice of z_1, it estimates what the range of likely electoral responses $\{\xi_t(z)\}$ will be. For each t it considers how its own choice, z_1, will affect coalition bargaining over policy outcomes and perquisites. It aggregates across t and chooses z_1 accordingly. Of course, with such a complex game there may be many Nash equilibria. However the uniqueness of the *PSNE* in the three-party game gives some hope that, at least generically, there will be a unique pure strategy Nash equilibrium in this game.

Suppose we consider the degenerate case examined in Section 8.5, where $t = 1$, so there is only one decisive coalition structure, namely the family comprising the three different two party coalitions. We have shown that the equilibrium outcome has expected value within the Pareto set. This suggests that in general the expected value of the equilibrium outcome under the game form g_ξ also lies inside the Pareto set of the parties. Current research seeks to substantiate these claims through computer simulation.

We may conjecture that a system of beliefs, regarding the electoral map ξ, and a system of known perquisites γ pertaining to government will give a unique outcome (in expectation) within the Pareto set of the parties' bliss points. Since this outcome is determined by ξ, it makes sense to ask what is the relationship between the expected policy outcome and the distribution of voters' preferences. If we let f be a distribution of voter bliss points, then we can define the electoral heart $\mathcal{H}(f)$ induced by this electoral distribution under plurality rule.

Implict in the discussion of Section 8.2 is the conclusion that decision-making under direct democracy, (with electoral prefrences represented by f) would lead to an outcome within $\mathcal{H}(f)$. Moreover for a large electorate, the results of Schofield and Tovey (1992) suggest that $\mathcal{H}(f)$ will be small relative to the electoral Pareto set. Politics is *efficient* if the expectation of the multiparty equilibrium policy outcome belongs to $\mathcal{H}(f)$.

8.7 Conclusion

The general model proposed here is that, in the pre-election environment, parties choose a Nash equilibrium to a game form g_ξ that represents their beliefs about electoral response, coalition policy outcome, and party perquisites from government. In stable regimes, parties would have a fairly good understanding of the game form, and in general might only need to consider two or three possible decisive structures. As we have emphasized, a key to the various possibilities will be whether or not one of the parties can present a core policy declaration.

In the context of such a model, there are a number of obvious questions:

1. How will the game change if new parties enter?

2. How will the equilibrium change if the pattern of perquisites is changed?

3. Is the pattern of convergence of "close" parties and divergence of "distant" parties that we observed in the three party case generic in the n-party case?

4. Under what conditions is this political game efficient?

5. Given that the game form g_ξ is a generalized version of the two party game, can we draw any general conclusion about the differences between two party and multiparty systems?

6. Can this model be used to describe how parties form in the first place?

7. In the post-election state $\{\mathcal{D}_t, z\}$, can the coalition probabilities $\{\rho_M\}$, coalition outcomes $\{z_M\}$ and perquisites $\{\gamma_M\}$ be determined endogenously? It is assumed in the model just presented that these are exogenously (or "historically") determined. As suggested, in stable regimes parties could form expectations of these outcomes. It is obvious however that in unstable situations parties will, in all likelihood, use a crude method of estimation to guess at post election events.

A useful goal for work on coalitions in multiparty systems would be to obtain information of the parameters of the general model (see for example the chapters in Part III of this volume).

The approach outlined here has attempted to integrate three rather different classes of political models. In Sections 8.2 and 8.3 we have considered coalition politics essentially as committee games, assuming that each party has well-defined policy preferences on X. The empirically-based discussion seems to give some credence to the notions of the core and heart. In Section 8.5 we considered again a committee of three parties, and proposed a model where parties make proposals, knowing that the outcome will be a lottery across coalitional compromises. Results on existence of a Nash equilibrium in such a game were presented. Finally Section 8.6 suggested how electoral risk could, in theory, be included in the model. Implicit in the model building is the assumption that parties do care about policy, and are not content simply to make declarations that maximize expected vote. A justification for this assumption is empirical work, just completed, on the multiparty system of Israel. Nixon *et al.* (1995) using data on electoral preferences and party declarations have statistically fitted a probabilistic voting model for the recent election of 1992. It turns out that the Nash equilibrium in the expected vote maximization model would have resulted in the coalescence of the parties into two groups, essentially based on the two large parties, Labor and Likud. Since the parties kept their separate identities and did not converge, this suggests that policy and electoral considerations are both important. The same argument would appear valid for all European multiparty polities.

There is a more general point to be made. On the one hand, all "Downsian" competitive electoral models, whether probabilistic or deterministic, exhibit "centripetal" tendencies towards convergence of party positions. On the other hand, the coalition model presented in Section 8.5 is based on cooperative, spatial voting theory. The very simple two dimensional model, presented in Section 8.5, exhibited a countervailing or "centrifugal" tendency towards divergence. The remaining fundamental research problem is to relate the nature of the electoral system (and other political institutional rules) to the balance attained between these two "forces" in a society, and to account for the formation of parties in different contexts. (See Cox 1990, for the effects of different electoral systems on these forces in the special case of a unidimensional policy space.)

It is possible, of course, that there can be situations where no "equilibrium" exists, or where it is unstable. Recent events in Russia, Italy and Japan, for example, may remind us that equilibria can persist for many years and then collapse. My view is that an integration of the analytical framework of cooperative game theory (as presented in section 8.2 for example) and of non-cooperative equilibrium theory provides the best hope for understanding the complexities of democratic politics.

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