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GENERALIZED SYMMETRY CONDITIONS AT A CORE POINT[1]

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Previous analyses have shown that if a point is to be a core of a majority rule voting game in Euclidean space, when preferences are smooth, then the utility gradients at the point must satisfy certain restrictive symmetry conditions. In this paper, these results are generalized to the case of an arbitrary voting rule, and necessary and sufficient conditions, expressed in terms of the utility gradients of "pivotal" coalitions, are obtained.

1. INTRODUCTION

IT IS NOW WELL KNOWN that if the set of alternatives can be represented as a subset of Euclidean space, and individual preferences are smooth, then the individual utility gradients at a point in the majority core must satisfy strong symmetry conditions (Plott (1967)). The necessity that these symmetry conditions be satisfied can be used to prove the generic nonexistence of core points in certain situations (McKelvey and Schofield (1986)). The same symmetry conditions can be used to show that if the majority rule core is empty, then it will generally be the case that voting trajectories can be constructed throughout the space.

This paper generalizes the Plott symmetry conditions to deal with arbitrary voting rules, obtaining restrictions on the gradients at a point which are necessary and sufficient for that point to be in the core. The generalized gradient restrictions that we identify show the central role of what we term the "pivotal" coalitions in determining when core points exist. Specifically, we define a coalition, M, to be *pivotal* in a subset L of the voters, if it is the case that whenever we partition $L-M$ into two subsets, at least one of these subsets, together with the members of M, constitutes a decisive coalition. Our symmetry conditions specify that for x to be a core point, the utility gradients of the members of any subset, L, of voters must satisfy the following condition: For every pivotal coalition M in L, the set of utility gradients which lie in the subspace spanned by those in M, must positively span 0 (the zero vector). Taking L to be the set of nonsatiated voters, it is easily shown that the Plott symmetry conditions for the existence of a majority core point are implied by this condition. The pivotal gradient condition can also be applied to get necessary conditions for a point to be in the constrained core, and hence for a point to be outside the cycle set of an arbitrary voting rule.

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2. DEFINITION AND NOTATION

We let $W \subseteq \boldsymbol{R}^w$ represent the set of *alternatives*. Let $N=\{1,2,\ldots,n\}$ be a finite set indexing *voters*. Let U denote the set of smooth, real valued functions on W, and let $u=(u_1,\ldots,u_n)\in U^n$, with u_i representing the *utility function* for voter i. Throughout this paper, we consider only a fixed $u\in U^n$, and call such a $u\in U^n$ a *smooth profile.*

For any binary relation $Q\subseteq W\times W$, we use the standard notation $xQy \Leftrightarrow (x,y)\in Q$. We write P_i for the binary relation on W defined by $xP_iy\Leftrightarrow u_i(x)>u_i(y)$, and for any $C\subseteq N$, write $P_C=\bigcap_{i\in C}P_i$.

We are given a set $\boldsymbol{D}$ of subsets $C\subseteq N$, called the set of *decisive coalitions*, which is assumed to satisfy: (a) $C\in \boldsymbol{D}$ and $C\subseteq C' \Rightarrow C'\in\boldsymbol{D}$ ($\boldsymbol{D}$ is monotonic); (b) $C\in\boldsymbol{D}\Rightarrow N-C\notin\boldsymbol{D}$ ($\boldsymbol{D}$ is proper). We can then define the *social order* $P\subseteq W\times W$ by

$$xPy \Leftrightarrow xP_Cy \quad \text{for some} \quad C\in\boldsymbol{D}.$$

For any binary relation, $Q\subseteq W\times W$, and $x\in W$, define $Q(x)=\{y\in X: yQx\}$, and write $Q^1(x)=Q(x)$. For any integer $j\geq 1$, define $Q^j(x)=\{y\in W: yQz$ for some $z\in Q^{j-1}(x)\}$. Then define $Q^*(x)=\bigcup_{j=1}^{\infty}Q^j(x)$. Also, for any $V\subseteq W$, and $Q\subseteq W\times W$, define $Q|_V=Q\cap(V\times V)$ to be the binary relation Q, restricted to V. We can then define the *core*, or *global optima set* to be the set of socially unbeaten alternatives in V:

$$GO(V,\boldsymbol{D})=\{x\in V: (P|_V)(x)=\varnothing\},$$

and the *local optima set* on V by

$$LO(V,\boldsymbol{D})=\{x\in V: x\in GO(V',\boldsymbol{D}) \text{ for some neighborhood } V' \text{ of } x \text{ in } V\}.$$

We define the *global cycle set* to be the set of points which are elements of a cycle in V, under the social order:

$$GC(V,\boldsymbol{D})=\{x\in V: x\in(P|_V)^*(x)\},$$

and the *local cycle set* by

$$LC(V,\boldsymbol{D})=\{x\in V: x\in GC(V',\boldsymbol{D}) \text{ for all neighborhoods, } V' \text{ of } x \text{ in } V\}.$$

When there is no fear of ambiguity we write $GO(V)$, $LO(V)$, etc. for these sets. We will also write $GO=GO(W)$, $LO=LO(W)$, etc. and call these the global or local optima sets with respect to $\boldsymbol{D}$. Clearly,

$$GO\subseteq LO \quad \text{and} \quad LC\subseteq GC.$$

3. CONSTRAINTS ON GRADIENTS AT A CORE POINT

In this section, we define the critical optimal set, $IO(W,\boldsymbol{D})$, give its relation to the global and local optima sets, and characterize this set in terms of conditions on the utility gradients of members of decisive coalitions.

For any $x \in W$, and $i \in N$, let $p_i(x) = \nabla u_i(x) \in \boldsymbol{R}^w$ represent voter i's utility gradient at the point x. For $C \subseteq N$, let

$$p_C(x) = \left\{ y \in \boldsymbol{R}^w \colon y = \sum_{i \in C} \alpha_i p_i(x),\ \alpha_i \geq 0\ \forall i \in C \text{ and } \exists i \in C \text{ s.t. } \alpha_i \neq 0 \right\}$$

be the semi-positive cone generated by $\{p_i(x) \colon i \in C\}$, and let

$$sp_C(x) = \left\{ y \in \boldsymbol{R}^w \colon y = \sum_{i \in C} \alpha_i p_i(x) \text{ with } \alpha_i \in \boldsymbol{R} \right\}$$

be the subspace spanned by $\{p_i(x) \colon i \in C\}$.[2]

We use the notation Int W to refer to the interior of W in the standard topology on $\boldsymbol{R}^w$, and write $\partial W = W \backslash \text{Int } W$ for the boundary of W. We also make the assumption that $W \subseteq \text{clos Int } W$ where clos means the closure in the topology on $\boldsymbol{R}^w$. This eliminates the possibility that W includes isolated points. Define the preference cone of coalition $C \subseteq N$ at x by

$$H_C^+(x) = \{y \in W \colon p_i(x) \cdot (y - x) > 0\ \forall i \in C\}.$$

Define the *critical* (or infinitesimal) *optima set* on $V \subseteq W$ with respect to $\boldsymbol{D}$ by

$$IO(V, \boldsymbol{D}) = \{x \in V \colon V \cap H_C^+(x) = \varnothing\ \forall C \in \boldsymbol{D}\}.$$

The critical optima set for $\boldsymbol{D}$ may be thought of as the analogue, for a social order, of the set of critical points of a smooth function. It is the set of points which, on the basis of "first derivative" information, are candidates for global optima. Thus the critical optima set contains the global optima set, but may also contain other points. We shall obtain necessary and sufficient conditions on the utility gradients at x for x to belong to $IO(W, \boldsymbol{D})$. Consequently these conditions will be necessary for a point to belong to the core. Under some conditions the critical and global optima sets coincide, and in this case, our conditions are necessary and sufficient for a point to belong to the core.

Say the smooth profile $(u_1, \dots, u_n)$ is *strictly pseudo-concave* iff $\forall i \in N$, and any $x, y \in W$ it is the case that $u_i(y) \geq u_i(x)$ implies that $p_i(x)(y - x) > 0$. More generally, say the preference profile is *semi-convex* iff $\forall i \in N$, and for any $x \in W$

$$\{y \in W \colon yP_i x\} \subseteq H_{\{i\}}^+(x).$$

It is easy to show that if the profile is strictly pseudo-concave then it is semi-convex in the above sense, and then $GO(W, \boldsymbol{D}) = IO(W, \boldsymbol{D})$.

LEMMA 1: (i) $GO(W) \subseteq LO(W) \subseteq IO(W, \boldsymbol{D})$. *Moreover if preferences are semi-convex, then these sets are identical.*

(ii) *If* $x \in \text{Int } W$ *then a necessary and sufficient condition for* $x \in IO(W, \boldsymbol{D})$ *is that* $0 \in \bigcap_{C \in \boldsymbol{D}} p_C(x)$.

[2] We use the convention that $sp_\varnothing(x) = \{0\}$, and $p_\varnothing(x) = \varnothing$.

PROOF: Using Taylor's Theorem, we can prove that if $H_C^+(x) \neq \varnothing$, for some $C \in \boldsymbol{D}$, then in any neighborhood V of x, $\exists y \in V$ such that yP_Cx (e.g., see Schofield (1984a), Lemma 4.19). Thus $x \notin IO(W, \boldsymbol{D})$ implies $x \notin LO(W)$ and hence $x \notin GO(W)$. When preferences are semi-convex, then for any $C \subseteq N$,

$$\{y \in W: yP_Cx\} \subseteq H_C^+(x).$$

Thus,

$$\begin{aligned} x \notin GO(W) &\Rightarrow H_C^+(x) \neq \varnothing \quad \text{for some} \quad C \in \boldsymbol{D} \\ &\Rightarrow x \notin IO(W, \boldsymbol{D}). \end{aligned}$$

(ii) From a standard argument (see, e.g., Schofield (1978 and 1983)) if $x \in \operatorname{Int} W$ then for any $C \subseteq N$,

$$H_C^+(x) = \varnothing \quad \text{iff} \quad 0 \in p_C(x).$$ *Q.E.D.*

Thus a necessary condition for $x \in \operatorname{Int} W \cap GO(W)$ is that $0 \in p_C(x)\ \forall C \in \boldsymbol{D}$. We now show that this latter condition is equivalent to a condition on *pivotal* rather than decisive coalitions.

4. SYMMETRY CONDITIONS FOR A CORE

In this section we define the notion of "pivotal" coalitions and use this notion to develop symmetry conditions, similar to the Plott (1967) symmetry condition for majority rule, which characterize $IO(W, \mathbf{D})$ for a fixed smooth profile, u.

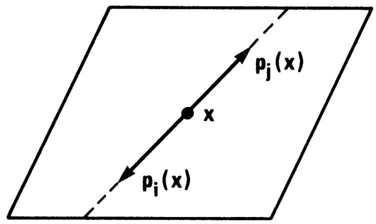
DEFINITION 1: Given any family $\boldsymbol{D}$ of subsets of N and any $L \subseteq N$, we define the set of *pivotal coalitions for* $\boldsymbol{D}$ *in* L, written $\boldsymbol{E}_L(\boldsymbol{D})$, as the set of all coalitions $M \subseteq L$ such that for every binary partition $\{C, D\}$ of $L - M$, either $M \cup C \in \boldsymbol{D}$ *or* $M \cup D \in \boldsymbol{D}$. *We write* $\boldsymbol{E}_L$ *for* $\boldsymbol{E}_L(\boldsymbol{D})$ when there is no danger of confusion.

It is easy to see that, since $\boldsymbol{D}$ is monotonic, so is $\boldsymbol{E}_L$; i.e., any superset of a pivotal coalition is also pivotal.

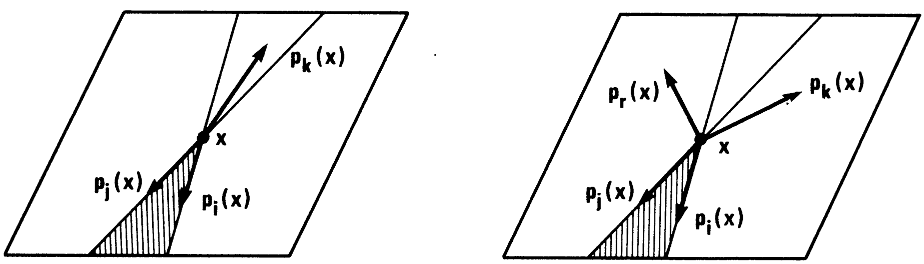
DEFINITION 2: Let $x \in W$. We say x satisfies the *pivotal gradient restrictions* (PGR) with respect to $\boldsymbol{D}$ iff, for every $L \subseteq N$ and every $M \in \boldsymbol{E}_L(\boldsymbol{D})$, $0 \in p_{M^*}(x)$, where $M^* = \{i \in L: p_i(x) \in sp_M(x)\}$.

We offer a loose interpretation of the above definition: Say that the pivotal coalition, $M \in \boldsymbol{E}_L$ is "blocked" if $0 \in p_{M^*}(x)$. If M is blocked, then there are some members of L, whose gradients lie in the same subspace as those of M, but *not* in the same half space. See Figure 1. Thus, the members of M^* cannot agree on any common direction to move. The PGR condition, then, simply specifies that every pivotal coalition, in every subset L of N, must be blocked in the above sense.

THEOREM 1: *If* $x \in \operatorname{Int} W$ *then a necessary and sufficient condition for* $x \in IO(W, \boldsymbol{D})$ *is that* x *satisfies PGR with respect to* $\boldsymbol{D}$.



(a) {i, j} blocked internally



(b) {i, j} blocked by other players

FIGURE 1—Examples of ways in which $\{i, j\}$ can be blocked.

PROOF: (i) Let $L \subseteq N$ and suppose, for some $M \in \boldsymbol{E}_L$, that $0 \notin p_{M^*}(x)$. Suppose that $\dim[sp_M(x)] = w$. Then $M^* = L$. But since $M \in \boldsymbol{E}_L$, then L contains some decisive coalition, C say. But then $0 \notin p_{M^*}(x)$ implies $0 \notin p_C(x)$, a contradiction. Suppose that $\dim[sp_M(x)] < w$. Then $\exists \beta \in \boldsymbol{R}^{\omega}$ with $\beta \cdot p_i(x) = 0$ for all $i \in M^*$, and $\beta \cdot p_i(x) \neq 0$ for all $i \in L - M^*$. Let $A = \{i \in L: \beta \cdot p_i(x) > 0\}$ and $B = \{i \in L: \beta \cdot p_i(x) < 0\}$. But since $M \in \boldsymbol{E}_L$, and $M^* \supseteq M$, we have $M^* \in \boldsymbol{E}_L$. Hence $M^* \cup A \in \boldsymbol{D}$ or $M^* \cup B \in \boldsymbol{D}$. Without loss of generality, assume $M^* \cup A \in \boldsymbol{D}$. Now if $0 \notin p_{M^*}(x)$, then by the separating hyperplane theorem, $\exists \alpha \in sp_{M^*}(x) = sp_M(x)$ with $\alpha \cdot p_i(x) > 0$ for all $i \in M^*$. Now pick $\delta \in \boldsymbol{R}^+$ with $(\beta + \delta\alpha) \cdot p_i(x) > 0$ for all $i \in A$ and set $\gamma = \beta + \delta\alpha$. Then $\gamma \cdot p_i(x) > 0$ for all $i \in M^* \cup A$. But then $0 \notin p_C(x)$ where $C = M^* \cup A \in \boldsymbol{D}$. By Lemma 1, $x \notin IO(\text{Int } W, \boldsymbol{D})$. Hence PGR is a necessary condition.[3]

(ii) To prove sufficiency, note that for any $M \in \boldsymbol{D}$ if we set $L = M$ then $M \in \boldsymbol{E}_L$ and $M^* = M$. Hence PGR$\Rightarrow$that $0 \in p_M(x)\ \forall M \in \boldsymbol{D}$. *Q.E.D.*

[3] The reader may wish to verify that, with the convention described in footnote 2, the proof of the theorem is valid in the case $M = \varnothing$.

COROLLARY 1: *PGR is a necessary condition for an interior point of W to belong to GO(W). Moreover, with semi-convex preferences the condition is also sufficient.*

PROOF: This follows directly from Theorem 1 together with Lemma 1.
Q.E.D.

5. APPLICATIONS TO GENERAL RULES

We now show how the PGR conditions can be applied to particular social choice functions, and how for majority rule, the conditions imply the Plott symmetry conditions.

Note that the PGR conditions specify symmetry conditions that must hold for *every* $L \subseteq N$. However, if $p_i(x)=0$ for some $i \in L$, then the PGR symmetry conditions are trivially satisfied for that L. Hence the most useful gradient restrictions are obtained by setting $L \subseteq \{i \in N: p_i(x) \neq 0\}$. In particular, a *necessary* condition for x to be a core point is that the PGR symmetry conditions be met for the set $L=\{i \in N: p_i(x) \neq 0\}$.

As an example, consider a *q-rule*, whose decisive coalitions are given by $\boldsymbol{D}=\{C \subseteq N: |C| \geq q\}$. The q-rule contains majority rule (with n odd or even) as a special case. Supra-majority rules of this kind have been studied by a number of writers (e.g., Ferejohn and Grether (1974), Greenberg (1979), Peleg (1978), Sloss (1973), Matthews (1980), Slutsky (1979). To obtain the core symmetry conditions for such a rule, assume that $q<n$ and define $e(n, q)=2q-n-1$. Then it is easy to verify (McKelvey and Schofield (1986)) that

(a) if $|L|=n$, then $\boldsymbol{E}_L=\{M \subseteq N: |M| \geq e(n, q)\}$;
(b) if $|L|=n-1$, then $\boldsymbol{E}_L=\{M \subseteq L: |M| \geq e(n, q)+1\}$.

Now, setting $L=\{i \in N: p_i(x) \neq 0\}$, we obtain necessary conditions for a point x to be a core point of a q-rule when no more than one person is satiated at x: Either no one is satiated at x, and all coalitions of size $e(n, q)$ are blocked, or one person is satiated at x, and all coalitions of size $e(n, q)+1$ (among the remaining individuals) are blocked. (Compare to Slutsky (1979).)

We now show how the Plott (1967) symmetry conditions for the existence of a majority rule core obtain as a special case of Theorem 1. Specifically, the Plott conditions deal with the case of majority rule when n is odd and when no two voters have common satiation points. The conditions specify the following:

CONDITION (PO): $p_j(x)=0$ for some $j \in N$, and for all $i \in N-\{j\}$, $\exists k \in N-\{i, j\}$ with $p_i(x)=-\alpha_k p_k(x)$ for some $\alpha_k>0$.

However, majority rule is a q-rule, with $q=(n+1)/2$, and $e(n, q)=0$, when n is odd. Setting $L=\{i \in N: p_i(x) \neq 0\}$ and using the characterization of the pivotal sets given above, it is easily verified that the pivotal gradient restrictions imply condition PO:

(a) If $|L| = n$, then $\boldsymbol{E}_L = \{M: |M| \geqslant 0\}$, so $\varnothing \in \boldsymbol{E}_L$. Since $sp_{\varnothing}(x) = \{0\}$ we see that $\varnothing^* = \varnothing$. But $p_{\varnothing}(x) = \varnothing$ contradicting the pivotal gradient restriction that $0 \in p_{\varnothing}(x)$. Hence $x \notin IO(W, \boldsymbol{D})$.

(b) If $|L| = n-1$, then $L = N - \{j\}$ for some $j \in N$ (i.e., $p_j(x) = 0$), and $\boldsymbol{E}_L = \{C \subseteq N - \{j\}: |C| \geqslant 1\}$. Hence, for all $i \in N - \{j\}$, $\{i\} \in \boldsymbol{E}_L$. Hence $0 \in p_{i^*}(x)$, which implies that $\exists k \in N - \{i, j\}$ with $p_i(x) = -\alpha_k p_k(x)$ for some $\alpha_k > 0$.

This gives Plott's theorem as an immediate corollary of Theorem 1.

COROLLARY 2: *Let P be majority rule, with n odd, and assume $x \in W$ satisfies $|\{i \in N: p_i(x) = 0\}| \leqslant 1$. Then $x \in GO \cap \mathrm{Int}\, W$ implies that condition PO is met.*

To show how Theorem 1 may be used in the general case, we let $n = 5$ and consider a social choice rule with the following decisive coalitions (we only list the minimal decisive sets): $\boldsymbol{D} = \{\{1, 2, 5\}, \{1, 3, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{4, 5\}\}$. Then the pivotal sets for $|L| \geqslant 4$ can be described as follows (we only list the minimal pivotal sets). Let $L_i = N - \{i\}$, and write $\boldsymbol{E}_{L_i} = \boldsymbol{E}_i$:

L	Pivotal Sets
N	$\boldsymbol{E}_N = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$
L_1	$\boldsymbol{E}_1 = \{\{2\}, \{3\}, \{4\}, \{5\}\}$
L_2	$\boldsymbol{E}_2 = \{\{1, 3\}, \{4\}, \{5\}\}$
L_3	$\boldsymbol{E}_3 = \{\{1, 2\}, \{4\}, \{5\}\}$
L_4	$\boldsymbol{E}_4 = \{\{1, 2\}, \{1, 3\}, \{5\}\}$
L_5	$\boldsymbol{E}_5 = \{\{2, 4\}, \{3, 4\}, \{2, 3\}\}$.

Thus, as above, setting $L = \{i \in N: p_i(x) \neq 0\}$, we obtain necessary conditions for x to be a core point if no more than one individual is satiated at x: Either no individual is satiated, and all coalitions in $\boldsymbol{E}_N$ are blocked, or individual i is satiated and all coalitions in $\boldsymbol{E}_i$ are blocked. Figure 2 illustrates how a core can occur at individual 5's ideal point in two dimensions, and Figure 3 illustrates how a core can occur at 5's ideal point in three dimensions. In these figures, we assume, for ease of illustration, that each player has a "Type I" or Euclidean

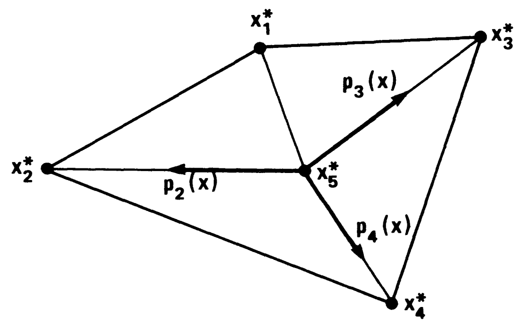


FIGURE 2—Example of core for $\boldsymbol{D}$ in two dimensions, at ideal point of individual 5.

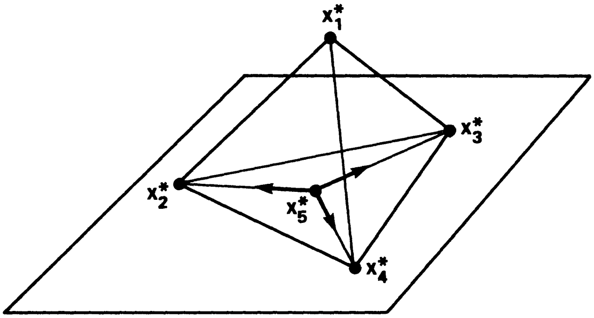


FIGURE 3*—Example of core for $\boldsymbol{D}$ in three dimensions, at ideal point of individual 5.

* The ideal point x_2^*, x_3^*, x_4^*, and x_5^*, are all in the subspace represented by the plane drawn. The ideal point x_1^*, may be off the plane.

preference of the type $u_i(x) = -1/2\|x - x_i^*\|^2$ where $\|\ \|$ is the standard Euclidean norm, and x_i^* is the ideal point of player i, where $p_i(x_i^*) = 0$.

In Figure 3, if x_5^* is the core, then this point must belong to the set A, defined to be the convex hull of $\{x_2^*, x_3^*, x_4^*\}$. Transversality arguments (McKelvey and Schofield (1986), and Smale (1973)) show that for an open dense set of profiles, the objects $\{x_5^*\}$ and A are respectively zero- and two-dimensional and do not intersect in $\boldsymbol{R}^3$. Thus the core in Figure 3 is "structurally unstable"; i.e., arbitrarily small perturbation will destroy the conditions for a core. On the other hand, in the two dimensional case of Figure 2, the pivotal gradient restrictions are robust under small perturbations and so the core is "structurally stable." Arguments of this sort can be used to establish the maximum dimension which will yield existence of structurally stable cores for arbitrary social choice functions. See McKelvey and Schofield (1986) and Schofield (1986)) for further discussion.

The results of Section 4 and these examples above are valid when the set of alternatives is unconstrained. Political institutions frequently impose feasibility constraints on social choice, and in the following section we show how these can be incorporated in more general pivotal gradient restrictions.

6. APPLICATIONS: CONSTRAINED CORES AND CYCLE SETS

We can use Theorem 1 to characterize points in a constrained core. We fix $x \in W$. Then for any $v \in \boldsymbol{R}^w$, define the v restriction on W by

$$W_v = \{y \in W: y \cdot v \geq x \cdot v\}.$$

Say that x is a *v constrained core,* whenever $x \in GO(W_v)$; i.e., x is a core in the constrained set W_v.

Another way of thinking of a constrained core is to introduce another voter, say voter "$n+1$," who has utility gradient $p_{n+1}(x) = v$, and who must be included in any winning coalition. Using this motivation, we define a new set $N_v = N \cup \{n+1\}$ of voters, and the corresponding set $\boldsymbol{D}_v$ of decisive coalitions by

$$\boldsymbol{D}_v = \{C \subseteq N_v: n+1 \in C \text{ and } C\backslash\{n+1\} \in \boldsymbol{D}\}.$$

Given $\boldsymbol{D}_v$, and any $L \subseteq N_v$, then as before $E_L(\boldsymbol{D}_v)$ is the set of pivotal coalitions for $\boldsymbol{D}_v$ in L. An easy argument shows that $IO(W_v, \boldsymbol{D}) = IO(W, \boldsymbol{D}_v)$. Then Theorem 1 immediately gives the following corollary.

COROLLARY 3: *If $x \in \text{Int } W$ then $x \in IO(W_v, \boldsymbol{D})$ iff x satisfies PGR with respect to $\boldsymbol{D}_v$.*

Applying Lemma 1 yields the following Corollary.

COROLLARY 4: *If $x \in \text{Int } W$ then a necessary condition for x to be a v-constrained core is that x satisfies PGR with respect to $\boldsymbol{D}_v$. If preferences are semi-convex, then the condition is sufficient.*

To illustrate, consider the case of majority rule with n odd. As we have noted, $\varnothing \in \boldsymbol{E}_N(\boldsymbol{D})$, so $\{n+1\} \in \boldsymbol{E}_{N_v}(\boldsymbol{D}_v)$. Hence, it follows from the pivotal gradient restrictions that there exists $k \in N$ with $p_k(x) = -\lambda v$ for some $\lambda \in \boldsymbol{R}$ with $\lambda > 0$. Now let $L = N_v \backslash \{k\}$. Then it follows that any set of the form $M = \{j, n+1\}$ is pivotal if $j \notin \{k, n+1\}$. It follows again, from the pivotal gradient conditions, that $0 \in p_{M^*}(x)$, where $M^* = \{i \in L: p_i(x) \in sp_M(x)\}$. In particular, it follows that $\exists i \in N \backslash \{j, k\}$ with $p_i(x) \in sp_M(x)$. But, if all gradients are nonzero, this is exactly the "joint symmetry" condition given by McKelvey (1979).

The symmetry properties for a majority rule constrained core which can be deduced from Corollary 4 are identical to those mentioned by Plott (1967). The corollary also shows how to obtain necessary symmetry properties at a constrained core for an arbitrary social order. Note also that Corollary 3 can be easily extended to the case where there exists a family of constraints at the point.

The notion of a constrained core is also helpful in characterizing the cycle set $LC(W)$. Define the *critical cycle set* (Schofield (1978)) written $IC(W)$ by $x \in IC(W)$ iff (i) $0 \notin p_C(x)$ for at least one $C \in \boldsymbol{D}$, and (ii) $\phi = \bigcap_{C \in \boldsymbol{D}(x)} p_C(x)$ where $\boldsymbol{D}(x) = \{C \in \boldsymbol{D}: 0 \notin p_C(x)\}$.

The critical cycle set bears the same relation to the local and global cycle sets as the critical optima bear to the local and global core points. It is the set of points which, on the basis of "first derivative information," are candidates for the cycle sets. Earlier results have shown that $IC(W)$ is open in W and

$$IC(W) \subseteq LC(W) \subseteq \text{clos } IC(W)$$

where clos $IC(W)$ is the closure of $IC(W)$ in W (Schofield (1978, 1984b)).

THEOREM 2: *$x \in (\text{Int } W) \backslash IC(W)$ iff there exists a vector $v_x \in \boldsymbol{R}^w \backslash \{0\}$ such that x satisfies PGR with respect to $\boldsymbol{D}_{v_x}$.*

PROOF: If $x \in (\text{Int } W) \backslash IC(W)$, then either (i) or (ii) in the definition of $IC(W)$ must fail. If (i) fails, then $0 \in \bigcap_{C \in \boldsymbol{D}} p_C(x)$, implying $x \in IO(W, \boldsymbol{D})$. Moreover, for any $v \in \boldsymbol{R}^w \backslash \{0\}$, $IO(W, \boldsymbol{D}) \subseteq IO(W_v, \boldsymbol{D})$. By Corollary 1, x must satisfy PGR with respect to $\boldsymbol{D}_v$. If (ii) fails, then there exists $v_x \in \boldsymbol{R}^w \backslash \{0\}$ such that, for all

$B \in \boldsymbol{D}$ either $0 \in p_B(x)$ or $-v_x \in p_B(x)$. Define $p_{n+1}(x) = v_x$. Now for any $B \subseteq N$, $B \in \boldsymbol{D} \Leftrightarrow B' = B \cup \{n+1\} \in \boldsymbol{D}_{v_x}$. Hence $0 \in p_B(x)$ or $-v_x \in p_B(x)$, $\forall B \in \boldsymbol{D} \Leftrightarrow 0 \in p_{B'}(x) \forall B' \in D_{v_x} \Leftrightarrow x$ satisfies *PRG* with respect to D_{v_x}, by Theorem 1. *Q.E.D.*

COROLLARY 5: If *LC*(*W*) *is empty then at every point in the interior of W, there exists a vector* $v_x \in \boldsymbol{R}^w$ *such that x satisfies PGR with respect to* $\boldsymbol{D}_{v_x}$.

PROOF: By previous results *LC*(*W*) is empty iff *IC*(*W*) is empty which implies that Int $W \cap IC(W)$ is empty. The result follows by Theorem 2. *Q.E.D.*

Early results by Cohen and Matthews (1980), McKelvey (1979), and Schofield (1983) only considered the cycle set for majority rule. Theorem 2, together with Corollary 5 and the comments following Corollary 4, give symmetry conditions which are necessary if a point is to lie outside the cycle set not just for majority rule but for an arbitrary social order.

Notice that Plott (1967, p 793) in his analysis of majority rule observed that a constraint could be represented by an "invisible" veto player. For an arbitrary social order the new player $(n+1)$ introduced in the proof of Corollary 3 and Theorem 2 has precisely the same function. This means effectively that $LC = \varnothing$ if and only if it is the case that, at each point x, there exists an "invisible" veto player i_x who in fact "represents" the social order.

Our work is related to that of Slutsky (1979) and Matthews (1980, 1982), who derive symmetry conditions for cores of alpha-majority rule and anonymous simple games, respectively. (Matthews also obtains conditions for constrained cores.) Their symmetry conditions give inequalities on the number of voters with gradients in opposing cones, in contrast to those here, which give classes of coalitions whose gradients must be "blocked." In addition to being applicable to arbitrary voting rules, our conditions have proven more useful than previous symmetry conditions in extending generic existence results for cores and cycle sets to more general voting rules (McKelvey and Schofield (1986)).

Finally, we note that the analysis of the previous sections can be extended to the case where W is a smooth manifold of dimension (dim (W)) equal to w.

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REFERENCES

COHEN, L., AND S. A. MATTHEWS (1980): "Constrained Plott Equilibria, Directional Equilibria and Global Cycling Sets," *Review of Economic Studies*, 47, 975–986.

FEREJOHN, J. A., AND D. M. GRETHER (1974): "On a Class of Rational Social Decision Procedures," *Journal of Economic Theory*, 8, 471–482.

GREENBERG, J. (1979): "Consistent Majority Rules over Compact Sets of Alternatives," *Econometrica*, 47, 285-297.
MATTHEWS, S. A. (1980): "Pairwise Symmetry Conditions for Voting Equilibria," *International Journal of Game Theory*, 9, 141-156.
——— (1982): "Local Simple Games in Public Choice Mechanisms," *International Economic Review*, 23, 623-645.
MCKELVEY, R. D. (1979): "General Conditions for Global Intransitivities in Formal Voting Models," *Econometrica*, 47, 1085-1112.
MCKELVEY, R. D., AND N. SCHOFIELD (1986): "Structural Instability of the Core," *Journal of Mathematical Economics*, 15, 179-198.
PELEG, B. (1978): "Consistent Voting Systems," *Econometrica*, 46, 153-161.
PLOTT, C. R. (1967): "A Notion of Equilibrium and its Possibility under Majority Rule," *American Economic Review*, 57, 787-806.
SCHOFIELD, N. (1978): "Instability of Simple Dynamic Games," *Review of Economic Studies*, 45, 575-594.
——— (1980): "Generic Properties of Simple Bergson-Samuelson Welfare Functions," *Journal of Mathematical Economics*, 7, 175-192.
——— (1983): "Generic Instability of Majority Rule," *Review of Economic Studies*, 50, 695-705.
——— (1984a): *Mathematical Methods in Economics*. New York: New York University Press.
——— (1984b): "Existence of Equilibrium on a Manifold," *Mathematics of Operations Research*, 9, 545-557.
——— (1986): "Existence of a 'Structurally Stable' Equilibrium for a Non-Collegial Voting Rule," *Public Choice*, 51, 267-284.
SLOSS, J. (1973): "Stable Outcomes in Majority Voting Games," *Public Choice*, 15, 19-48.
SLUTSKY, S. (1979): "Equilibrium under α-Majority Voting," *Econometrica*, 47, 1113-1125.
SMALE, S. (1973): "Global Analysis and Economics I: Pareto Optimum and a Generalization of Morse Theory," in *Dynamical Systems*, ed. by M. Peixoto. New York: Academic Press.