SOCIAL CHOICE (6.154.11)

Norman Schofield
Center in Political Economy
Washington University in Saint Louis, MO 63130
USA
Phone: 314 935 4774
Fax: 314 935 4156
E-mail: schofield.norman@gmail.com
Key words: Impossibility Theorem, Cycles, Nakamura Theorem, Voting..

Contents

Summary
Arrows Impossibility implies that any social choice procedure that is rational and satisfies the Pareto condition will exhibit a dictator, an individual able to control social decisions. If instead all that we require is the procedure gives rise to an equilibrium, core outcome, then this can be guaranteed by requiring a collegium, a group of individuals who together exercise a veto. On the other hand, any voting rule without a collegium is classified by a number,v, called the Nakumura number. If the number of alternatives does not exceed v, then an equilibrium can always be guaranteed. In the case that the alternatives comprise a subset of Euclden spce, of dimension w, then an equilibrium can be guaranteed as long as $w \leq v - 2$. In general, however, majority rule has Nakumura number of 3, so an equilibrium can only be guaranteed in one dimension.

1. Introduction

1.1. Rational Choice

A fundamental question that may be asked about a political, economic or social system is whether it is responsive to the wishes or opinions of the members of the society and, if so, whether it can aggregate the conflicting notions of these individuals in a way which is somehow rational. More particularly, is it the case, for the kind of configuration of preferences that one might expect, that the underlying decision process gives rise to a set of outcomes which is natural and stable, and more importantly, "small" with respect to the set of all possible outcomes? If so, then it may be possible to develop a theoretical or "causal" account of the relationship between the nature of the decision process, along with the pattern of preferences, and the behavior of the social system under examination. For example, microeconomic theory is concerned with the analysis of a method of preference aggregation through the market which results, under certain conditions at least, in a particular distribution of prices for commodities and labor, and thus income. The motivation for this venture is to mimic to a degree the ability of some disciplines in natural science to develop causal models which tie initial conditions of the physical system to a small set of predicted outcomes. The theory of democratic pluralism is to a large extent based on the assumption that the initial conditions of the political system are causally related to the essential properties of the system. That is to say it is assumed that the interaction of cross-cutting interest groups in a democracy leads to an "equilibrium" outcome that is natural in the sense of balancing the divergent interests of the members of the society. One aspect of course of this theoretical assumption is that it provides a method of legitimating the consequences of political decision making.

The present work directs attention to those conditions under which this assumption may be regarded as reasonable. For the purposes of analysis it is assumed that individuals may be represented in a formal fashion by preferences which are "rational" in some sense. The political system in turn is represented by a social choice mechanism, such as, for example, a voting rule. It is not assumed that society may in fact be represented in just this fashion. Rather the purpose is to determine whether such a skeleton of a political system is likely to exhibit an equilibrium. The method that is adopted is to classify all such abstract political systems. It turns out that a pure social equilibrium is very rare. This seems to suggest that if "real world" political systems are in fact in equilibrium, then what creates this equilibrium is not at all to do with what might be thought to be the initial conditions – that is to say preferences and institutional rules. Rather it must be that there are other aspects of the political system, which together with preferences and formal rules, are sufficient to generate equilibrium. It might be appropriate to use the term abstract political economy for a representation of the polity which does incorporate these further additional features.

Social choice is the theoretic discipline which is concerned with the analysis of systems of choice where the primitives are precisely preferences and rules. The development of the theory has often taken the form of focussing on certain equilibrium properties of choice mechanisms, only to find that the conditions under which they are satisfied are rare in some sense. It is useful to give a brief overview of this development of social choice, partly because there is a parallel with the structure of the current work, but also because it indicates how earlier results can be fitted into the general classification theorem for social choice mechanisms.

1.2. The Theory of Social Choice

Social choice theory assumes that each individual i in a society $N = \{1, \ldots, n\}$ is characterized by a "rational" preference relation p_i, so that the society is represented by a profile of preference relations, $p = (p_i, \ldots, p_n)$, one for each individual. Let the set of possible alternatives be $W = \{x, y, \ldots\}$. If person i prefers x to y then write $(x, y)\varepsilon p_i$, or more commonly xp_iy. The social mechanism or *preference function*, σ, translates any profile p into a preference relation $\sigma(p)$. The point of the theory is to examine conditions on σ which are sufficient to ensure that whatever "rationality properties" are held by the individual preferences, then these same properties are held by $\sigma(p)$. Arrow's Impossibility Theorem (1951) essentially showed that if the rationality property under consideration is that preference be a weak order then σ must be dictatorial. To see what this means, let R_i be the weak preference for i induced from p_i. That is to say xR_iy if and only if it is not the case that yp_ix. Then p_i is called a *weak order* if and only if .R_i is *transitive* i.e., if xR_iy and yR_iz for some x, y, z in W, then xR_iz. Arrow's theorem effectively demonstrated that if $\sigma(p)$ is a weak order whenever every individual has a weak order preference then there must be some dictatorial individual i, say, who is characterized by the ability to enforce every social choice..

It was noted some time afterwards that the result was not true if the conditions of the theorem were weakened. For example, the requirement that $\sigma(p)$ be a weak order means that "social indifference" must be transitive. If it is only required that strict social preference be transitive, then there can indeed be a non-dictatorial social preference mechanism with this weaker rationality property (Sen, 1970). To see this, suppose σ is defined by the *strong Pareto rule*: $x\sigma(p)y$ if and only if there is no individual who prefers y to x but there is some individual who prefers x to y. It is evident that σ is non-dictatorial. Moreover if each p_i is transitive then so is $\sigma(p)$. However, $\sigma(p)$ cannot be a weak order. To illustrate this, suppose that the society consists of two individuals $\{1, 2\}$ who have preferences

1	2
x	y
z	x
y	z

This means xp_1zp_1y etc. Since $\{1, 2\}$ disagree on the choice between x and y and also on the choice between y and z both x, y and y, z must be socially indifferent. But then if $\sigma(p)$ is to be a weak order, it must be the case that x and z are indifferent. However, $\{1, 2\}$ agree that x is superior to z, and by the definition of thestrong Pareto rule, x must be chosen over z. This of course contradicts transitivity of social indifference.

A second criticism due to Fishburn (1970) was that the theorem was not valid in the case that the society was infinite. Indeed since democracy often involves the aggregation of preferences of many millions of voters the conclusion could be drawn that the theorem was more or less irrelevant.

However, three papers by Gibbard (1969), Hanssen (1976) and Kirman and Sondermann (1972) analyzed the proof of the theorem and showed that the result on the existence of a dictator was quite robust. Section 2.3 essentially parallels the proof by Kirman and Sondermann. The key notion here is that of a *decisive coalition*: a coalition M is decisive for a social choice function, σ, if and only if xp_iy for all i belonging to M for the profile p implies $x\sigma(p)y$. Let $\mathbb{D}_\sigma$ represent the set of decisive coalitions defined by σ. Suppose now that there is some coalition, perhaps the whole society N, which is decisive. If σ preserves transitivity (i.e., $\sigma(p)$ is transitive) then the intersection of any two decisive coalitions must itself be decisive. The intersection of all decisive coalitions must then be decisive: this smallest decisive coalition is called an *oligarchy*. The oligarchy may indeed consist of more than one individual. If it comprises the whole society then the rule is none other than the Pareto rule. However, in this case every individual has a veto. A standard objection to such a rule is that the set of chosen alternatives may be very large, so that the rule is effectively indeterminate. Suppose the further requirement is imposed that $\sigma(p)$ always be a weak order. In this case it can be shown that for any coalition M either M itself or its complement $N\backslash M$ must be decisive. Take any decisive coalition A, and consider a proper subset B say of A. If B is not decisive then $N\backslash B$ is, and so $A \cap (N\backslash B) = A\backslash B$ is decisive. In other words every decisive coalition contains a strictly smaller decisive coalition. Clearly, if the society is finite then some individual is the smallest decisive coalition, and consequently is a dictator. Even in the case when N is infinite, there will be a smallest "invisible" dictator. It turns out, therefore, that reasonable and relatively weak rationality properties on σ impose certain restrictions on the class $\mathbb{D}_\sigma$ of decisive coalitions. These restrictions on $\mathbb{D}_\sigma$ do not seem to be similar to the characteristics that political systems display. As a consequence these first attempts by Sen and Fishburn and others to avoid the Arrow Impossibility Theorem appear to have little force.

A second avenue of escape is to weaken the requirement that $\sigma(p)$ always be transitive. For example a more appropriate mechanism might be to make a choice from W of all those *unbeaten* alternatives. Then an alternative x is chosen if and only if there is no other alternative y such that $y\sigma(p)x$. The set of unbeaten alternatives is also called the *core* for $\sigma(p)$, and.is defined by

$$Core(\sigma, p) = \{x \varepsilon W : y\sigma(p)x \text{ for no } y\varepsilon W\}.$$

In the case that W is finite the existence of a core is essentially equivalent to the requirement that $\sigma(p)$ be *acyclic* (Sen, 1970). Here a preference, p, is called acyclic if and only if whenever there is a chain of preferences

$$x_0px_1px_2p\cdots px_r$$

then it is not the case that x_rpx_0.

However, acyclicity of σ also imposes a restriction on $\mathbb{D}_\sigma$. Define the *collegium* $\kappa(\mathbb{D}_\sigma)$ for the family $\mathbb{D}_\sigma$ of decisive coalitions of σ to be the intersection (possibly empty) of all the decisive coalitions. If the collegium is empty then it is always possible to construct a "rational" profile p such that $\sigma(p)$ is cyclic (Brown, 1973). Therefore, a necessary condition for σ to be acyclic is that σ exhibit a non-empty collegium. We say σ is *collegial* in this case. Obviously, if the collegium is large then the rule is indeterminate, whereas if the collegium is small the rule is almost dictatorial.

A third possibility is that the preferences of the members of the society are restricted in some way, so that natural social choice functions. such as majority rule will be "well behaved". For example, suppose that the set of alternatives is a closed subset of a single dimensional "left-right" continuum. Suppose further that each individual i has *convex* preference on W, with a most preferred point (or bliss point) x_i, say.[1] Then a well-known result by Black (1958) asserts that the

[1]Convexity of the preference p just means that for any y the set $\{x : xpy\}$ is convex. A natural preference to use is Euclidean preference defined by xp_iy if and only if $||x - x_i|| < ||y - x_i||$, for some bliss point, x_i, in W, and norm $||-||$ on W. Clearly Euclidean preference is convex.

core for majority rule is the the median most preferred point. On the other hand, if preferences are not convex, then as Kramer and Klevorick (1974) demonstrated, the social preference relation $\sigma(p)$ can be cyclic, and thus have an empty core. However, it was also shown that there would be a *local core* in the one dimensional case. Here a point is in the local core, $LO(\sigma, p)$, if there is some neighborhood of the point which contains no socially preferred points.

The idea of preference restrictions sufficient to guarantee the existence of a majority rule core was developed further in a series of papers by Sen (1966), Inada (1969) and Sen and Pattanaik (1969). However, it became clear, at least in the case when W had a geometric form, that these preference restrictions were essentially only applicable when W was one dimensional.

To see this suppose that there exist a set of three alternatives $X = \{x, y, z\}$ in W, and three individuals $\{1, 2, 3\}$ in N whose preferences on X are:

1	2	3
x	y	z
y	z	x
z	x	y

The existence of such a *Condorcet Cycle* is in contradiction to all the preference restrictions. If a profile p on W, containing such a Condorcet Cycle, can be found then there is no guarantee that $\sigma(p)$ will be acyclic or exhibit a non-empty core. Kramer (1973) effectively demonstrated that if W were two dimensional then it was always possible to construct convex preferences on W such that p contained a Condorcet Cycle. Kramer's result, while casting doubt on the likely existence of the core, did not, however, prove that it was certain to be empty. On the other hand an earlier result by Plott (1967) did show that when the W was a subset of Euclidean space, and preference convex and smooth, then, for a point to be the majority rule core, the individual bliss points had to be symmetrically distributed about the core. These Plott symmetry conditions are sufficient for existence of a core when n is either odd or even, but are necessary when n is odd. The "fragility" of these conditions suggested that a majority rule core was unlikely in some sense in high enough dimension (McKelvey and Wendell, 1976). It turns out that these symmetry conditions are indeed fragile in the sense of being "non-generic" or atypical..

An article by Tullock (1967) at about this time argued that even though a majority rule core would be unlikely to exist in two dimensions, nonetheless it would be the case that cycles, if they occurred, would be constrained to a central domain in the Pareto set (i.e., within the set of points unbeaten under the Pareto rule).

By 1973, therefore, it was clear that there were difficulties over the likely existence of a majority rule core in a geometric setting. However, it was not evident how existence depended on the number of dimensions. Later results by McKelvey and Schofield (1987) and Saari (1997) indicate how the behavior of a general social choice rule is dependent on the dimensionality of the space of alternatives.

1.3. Restrictions on the Set of Alternatives

One possible way of indirectly restricting preferences is to assume that the set of alternatives, W, is of finite cardinality, r, say. As Brown (1973) showed, when the social preference function σ is not collegial then it is always possible to construct an acyclic profile such that $\sigma(p)$ is in fact cyclic. However, as Ferejohn and Grether (1974) proved, to be able to construct such a profile it is necessary that W have a sufficient cardinality. These results are easier to present in the case of a *voting rule* σ . Such a rule,σ, is determined completely by its decisive coalitions,$\mathbb{D}_\sigma$.That is to say:

$$x\sigma(p)y \text{ if and only if } xp_iy \text{ for every } i\varepsilon M, \text{ for some } M\varepsilon\mathbb{D}_\sigma.$$

An example of a voting rule is a $q-rule$,written σ_q, and the decisive coalitions for σ_q are defined to be

$$\mathbb{D}_q = \{M \subset N : \ |M| \geq q.$$

Clearly if $q < n$ then $\mathbb{D}_q$ has an empty collegium. Ferejohn and Grether (1974) showed that if

$$q > (\frac{r-1}{r})n \text{ where } |W| = r$$

then no acyclic profile , p be constructed so that $\sigma(p)$ was cyclic. Conversely if $q \leq (\frac{r-1}{r})n$ then such a profile could certainly be constructed. Another way of expressing this is that a *q-rule* σ is acyclic for all acyclic profiles if and only if $|W| < \frac{n}{n-q}$. Note that we assume that $q < n$.

Nakamura (1979) later proved that this result could be generalized to the case of an arbitrary social preference function. The result depends on the notion of a *Nakamura number* $v(\sigma)$ for σ. Given a non-collegial family $\mathbb{D}$ of coalitions, a member M of $\mathbb{D}$ is *minimal decisive* if and only if M belongs to $\mathbb{D}$, but for no member i of M does $M \backslash \{i\}$ belong to $\mathbb{D}$. If $\mathbb{D}'$ is a subfamily of $\mathbb{D}$ consisting of minimal decisive coalitions, and moreover $\mathbb{D}'$ has an empty collegium then call $\mathbb{D}'$ a *Nakumura subfamily* of $\mathbb{D}$. Now consider the collection of all Nakamura subfamilies of $\mathbb{D}$. Since N is finite these subfamilies can be ranked by their cardinality. Define $v(\mathbb{D})$ to be the cardinality of the smallest Nakamura subfamily, and call $v(\mathbb{D})$ the *Nakamura number* of $\mathbb{D}$. Any Nakamura subfamily $\mathbb{D}'$, with cardinality $|\mathbb{D}'| = v(\mathbb{D})$, is called a *minimal non-collegial subfamily.* When σ is a social preference function with decisive family $\mathbb{D}_\sigma$ define the Nakamura number $v(\sigma)$ of σ to be equal to $v(\mathbb{D}_\sigma)$. More formally

$$v(\sigma) = \min\{|\mathbb{D}'| \colon \mathbb{D}' \subset \mathbb{D} \text{ and } \kappa(\mathbb{D}') = \Phi\}.$$

In the case that σ is collegial then define

$$v(\sigma) = v(\mathbb{D}_\sigma) = \infty \text{ (infinity)}.$$

Nakamura showed that for any voting rule ,σ, if W is finite, with $|W| < v(\sigma)$ then $\sigma(p)$ must be acyclic whenever p is an acyclic profile. On the other hand, if σ is a social preference function and $|W| \geq v(\sigma)$ then it is always possible to construct an acyclic profile on W such that $\sigma(p)$ is cyclic. Thus the cardinality restriction on W which is necessary and sufficient for σ to be acyclic is that $|W| < v(\sigma)$. To relate this to Ferejohn-Grether's result for a *q-rule*, define $v(n,q)$ to be the largest integer such that $v(n,q) < \frac{q}{n-q}$. It is an easy matter to show that when σ_q is a q-rule then

$$v(\sigma_{q)} = 2 + v(n,q).$$

The Ferejohn-Grether restriction $|W| < \frac{n}{n-q}$ may also be written

$$|W| < 1 + \frac{q}{n-q}$$

which is the same as

$$|W| < v(\sigma_{q)}.$$

Thus Nakamura's result is a generalization of the earlier result on q-rules.

The interest in this analysis is that Greenberg (1979) showed that a core would exist for a q-rule as long as preferences were convex and the choice space, W, was of restricted dimension. More precisely suppose that W is a compact[2] , convex subset of Euclidean space of dimension w, and suppose each individual preference is continuous[3] and convex. If $q > (\frac{w}{w+1})n$ then the core of $\sigma(p)$ must be non-empty, and if $q \leq (\frac{w}{w+1})n$ then a convex profile can be constructed such that the core is empty. From a result by Walker (1977) the second result also implies, for the constructed profile p that $\sigma(p)$ is cyclic. Rewriting Greenberg's inequality it can be seen that the necessary and sufficient dimensionality condition (given convexity and compactness) for the existence of a core and the non-existence of cycles for a q-rule, σ_q, is that $\dim(W) \leq v(n,q)$ where $\dim(W) = w$ is the dimension of W.

Since

$$v(\sigma_{q)} = 2 + v(n,q).$$

where $v(\sigma_{q)}$ is the Nakamura number of the $q-rule$, this suggests that for an arbitrary non-collegial voting rule σ there is a *stability dimension*, namely $v^*(\sigma) = v(\sigma) - 2$, such that $\dim(W) \leq v^*(\sigma)$ is a necessary and sufficient condition for the existence of a core and the non-existence of cycles.

An important procedure in this proof is the construction of a *representation* ϕ for an arbitrary social preference function. Let $\mathbb{D} = \{M_1, ..., M_v\}$ be a minimal non-collegial subfamily for σ. Note that $\mathbb{D}$ has empty collegium and cardinality $v(\sigma) = v$. Then σ can be represented by a $(v-1)$ dimensional simplex Δ in $\mathbb{R}^{v-1}$. Moreover, each of the v faces of this simplex can be identified with one of the v coalitions in $\mathbb{D}$. Each proper subfamily $\mathbb{D}_t = \{.., M_{t-1}, M_{t+1}, ..\}$ has a non-empty collegium, $\kappa(\mathbb{D}_t)$, and each of these can be identified with one of the vertices of Δ. To each $i\ \varepsilon\kappa(\mathbb{D}_t)$ we can assign a preference p_i, for i=1,...$v\}$ on a set $x = \{x_{1,}x_2, \ldots, x_v\}$ giving a *permutation* profile

$\kappa(\mathbb{D}_1)$	$\kappa(\mathbb{D}_2)$	$\ldots$	$\kappa(\mathbb{D}_v)$
x_1	x_2		x_v
x_2	x_3		x_1
.	.		. .
.	.		.
.	.		.
x_v	x_1	$\ldots$	x_{v-1}

From this construction it follows that

$$x_1\sigma(p)x_2 \cdots \sigma(p)x_v\sigma(p)x_{1.}$$

Thus whenever W has cardinality at least v, then it is possible to construct a profile p such that $\sigma(p)$ has a permutation cycle of this kind. This representation theorem is used in Chapter 4 to prove Nakamura's result and to extend Greenberg's Theorem to the case of an arbitrary rule.

The principal technique underlying Greenberg's theorem is an important result due to Fan (1961). Suppose that W is a compact convex subset of a topological vector space, and suppose P is a correspondence from W into itself which is convex and continuous[4]. Then there exists an

[2]Compactness just means the set is closed and bounded.

[3]The continuity of the preference, p, that is required is that for each $x\varepsilon W$, the set $\{y\varepsilon W : xpy\}$ is open in the topology on W.

[4]The continuity of the preference, P, that is required is that for each $x\varepsilon W$, the set $P^{-1}(x)$ =$\{y\varepsilon W : x\varepsilon P(y)\}$ is open in the topology on W.

"equilibrium" point x in W such that $P(x)$ is empty. In the case under question if each individual preference, p_i, is continuous, then so is the preference correspondence P associated with $\sigma(p)$. Moreover, if W is a subset of Euclidean space with dimension no greater than $v(\sigma)-2$, then using Caratheodory's Theorem it can be shown that P is also convex. Then by Fan's Theorem, P must have an equilibrium in W. Such an equilibrium is identical to the core,$Core(\sigma, p)$.

On the other hand, suppose that $\dim(W) = v(\sigma) - 1$. Using the representation theorem, the simplex Δ representing σ can be embedded in W. Let $Y = \{y_1, \ldots, y_v\}$ be the set of vertices of Δ. As above, let $\{\kappa(\mathbb{D}_t) : t = 1, ..., v\}$ be the various collegia. Each player $i\varepsilon\kappa(\mathbb{D}_t)$, is associated with the vertex y_t and is assigned a "Euclidean" preference of the form xp_iz if and only if $||x - y_i|| < ||z - y_i||$. In a manner similar to the situation with W finite, it is then possible to show, with the profile p so constructed, that for every point z in W there exists x in W such that $x\sigma(p)z$. Thus the core for $\sigma(p)$ is empty and $\sigma(p)$ must be cyclic. In the case that W is compact, convex, and preference is continuous and convex, then a necessary and sufficient condition for the existence of the core, and non-existence of cycles is that $\dim(W) \leq v^*(\sigma)$, where $v^*(\sigma) = v(\sigma) - 2$ is called the *stability dimension*. This result was independently obtained by Schofield (1984a,b) and Strnad (1985) and will be called the Schofield Strnad Theorem.

This results on the Nakamura number can be extended by showing that even with non-convex preference, a *"critical" core* called $IO(\sigma, p)$, which contains the *local core*, $LO(\sigma, p)$, will exist as long as $\dim(W) \leq v^*(\sigma)$. It is an easy matter to show that for majority rule $v^*(\sigma) \geq 1$, and so this gives an extension of the Kramer-Klevorick (1974) Theorem.

1.4. Structural Stability of the Core

Although the σ-core cannot be guaranteed in dimension $v^*(\sigma) + 1$ or more, none the less it is possible for a core to exist in a "structurally stable" fashion. We now assume that each preference p_i can be represented by a smooth utility function $u_i \colon W \to \mathbb{R}$. As before, this means simply that

$$xp_iy \text{ if and only if } u_i(x) > u_i(y).$$

A smooth profile for the society N is a differentiable function

$$u = (u_1, \ldots, u_n) \colon W \to \mathbb{R}^n.$$

We assume in the following analysis that W is compact, and let $U(W)^N$ be the space of all such profiles endowed with the Whitney C^2-topology (Golubitsky and Guellemin, 1973; Hirsch, 1976). Essentially two profiles u^1 and u^2 are close in this topology if all values and the first and second derivatives are close.

Restricting attention to smooth utility profiles whose associated preferences are convex gives the space $U_{con}(W)^N$ Say that the core $Core(\sigma, u)$ for a rule σ is *structurally stable* (in $U_{con}(W)^N$) if $Core(\sigma, u)$ is non-empty and there exists a neighborhood V of u in $U_{con}(W)^N$ such that $Core(\sigma, u')$ is non-empty for all u' in V. To illustrate, if $Core(\sigma, u)$ is non-empty but not structurally stable then an arbitrary small perturbation of u is sufficient to destroy the core by rendering it empty.

By a previous result if $\dim(W) \leq v^*(\sigma)$ then $Core(\sigma, u)$ is non-empty for every smooth, convex profile, and thus this dimension constraint is sufficient for $Core(\sigma, u)$ to be structurally stable.

It had earlier been shown by Rubinstein (1979) that the set of continuous profiles such that the majority rule core is non-empty is in fact a nowhere dense set in a particular topology on profiles, independently of the dimension.. However, the perturbation involved deformations induced by creating non-convexities in the prefered sets. Thus the construction did not deal with the question of structural stability in the topological space $U_{con}(W)^N$

Results by McKelvey and Schofield (1987) and Saari (1997) which indicate that for any $q-$rule, σ_q, there is an *instability dimension* $w(\sigma_q)$. If $\dim(W) \geq w(\sigma_q)$ and W has no boundary then the σ_q-core is empty for a dense set of profiles in $U(W)^N$. This immediately implies that the core cannot be structurally stable, so any sufficiently small perturbation in $U_{con}(W)^N$ will destroy the core. The same result holds if W has a non-empty boundary but $\dim(W) \geq w(\sigma_q) + 1$. To prove this result, necessary and sufficient conditions, for a point to belong to the core, are examined.

The easiest case to examine is where the core, $Core(\sigma, u)$, is characterized by the property that exactly one individual has a bliss point at the core. In this case we use the term *Bliss Core* and denote this by $BCore(\sigma, u)$. Theorem 6.1.3 then shows that if $x \varepsilon\ BCore(\sigma, u)$ there is a coalition R with $|R| = 2q - n + 1$ such that the direction gradients, at x, of the utility functions of the members of R, must be semi-positively dependent. Thus implies that x must belong to a "singularity manifold" $\wedge(R, u)$. In the case that $\dim(W) \geq |R|$ then the Thom Transversality Theorem (Golubitsky and Guellemin, 1973;Hirsch, 1976) implies that $\dim \wedge(R, u) \leq |R| - 1$ "almost always" (i.e.,generically, or for a dense set of profiles in the space $U_{con}(W)^N$). Moreover, if $x = x_1$, the bliss point of player 1,then (if is in the interior of W) it must be a critical point of $u_{1,}$ and we can assume $x_1\ \varepsilon \wedge(1, u)$,the singularity manifold of u_1. This generically has dimension 0.. Finally, if $\dim(W) \geq 2q - n + 1$ then the intersection of $\wedge(M, u)]$ and $\wedge(1, u)$ has dimension

$$\dim[\wedge(M, u)] \cap \wedge(1, u)] < 0$$

generically. This implies that $BCore(\sigma_q, u)$ is generically empty. This suggests that the instability dimension satisfies $w(\sigma_q) = 2q - n + 1$.

Saari (1997) extended this result in two directions, by showing that if $\dim(W) \leq 2q - n$ then $BCore(\sigma_q, u)$ could be structurally stable, Moreover, he was able to compute the instability dimension for the case of a non-bliss core, when no individual has a bliss point at the core.

For example, with majority rule $(2q - n + 1)$ is two or three depending on whether n is odd or even. For n odd, both bliss and non-bliss cores cannot occur generically in two or more dimensions, since the Plott (1967) symmetry conditions cannot be generically satisfied. On the other hand, when $(n, q) = (4, 3)$, the Nakamura number is four, and hence a core will exist in two dimensions. Indeed, both bliss cores and non-bliss cores can occur in a structurally stable fashion. However, in three dimensions the cycle set is contained in, but fills the Pareto set. For $(n, q = (6, 4)$ and all other majority rules with n even and $n \geq 6$, a structurally stable bliss-core can occur in two dimensions. In three dimensions the core cannot be structurally stable and the cycle set need not be constrained to the Pareto set (in contradiction to Tullock's hypothesis).

2. Social Choice

2.1. Preference Relations

Social Choice is concerned with a fundamental question in political or economic theory: is there some process or rule for decision making which can give consistent social choices from consistent individual choices?

In this framework denote by W a *universal set of alternatives*. Members of W will be written x, y etc. The *society* is denoted by N, and the individuals in the society are callen $1, \ldots, i, \ldots, j, \ldots n$. The values of an individual i are represented by a preference relation p_i on the set W. Thus $xp_{iy}is$ taken to mean that individual i prefers alternative x to alternative y. It is also assumed that each $p_i is$ *strict*, in the way to be described below. The rest of this section considers the abstract properties of a preference relation p on W.

Definition 2.1.1. *A strict preference relation p on W* is

(i) *Irreflexive* : for no $x \varepsilon W$ does xpx ;

(ii) *Asymmetric* :for any $x, y, \varepsilon W; xpy \Rightarrow$ not(ypx).

The strict preference relations are regarded as fundamental primitives in the discussion. No attempt is made to determine how individuals arrive at their preferences, nor is the problem considered how preferences might change with time. A preference relation p may be *represented* by a utility function.

Definition 2.1.2. A preference relation p is *representable* by a utility function

$$u\colon W \to \mathbb{R} \text{ for any } x, y \varepsilon W; xpy \Leftrightarrow u(x) > u(y).$$

Alternatively, p is representable by u whenever

$$\{x\colon u(x) > u(y)\} = \{x\colon xpy\}\text{for any } y\varepsilon W.$$

If both $u^1, u^2\colon W \to \mathbb{R}$ represent p then write $u^1 \sim u^2$. The equivalence class of real valued functions which represents a given p is called an *ordinal utility function* for p, and may be written u_p. If p can be represented by a continuous (or smooth) utility function, then we may call p continuous (or smooth).

By some abuse of notation we shall write:

$$u_p(x) > u_p(y)$$

to mean that for any $u\colon W \to R$ which represents p it is the case that $u(x) > u(y)$. We also write $u_p(x) = u_p(y)$ when for any u representing p,it is the case that $u(x) = u(y)$.

From the primitive strict preference relation p define two new relations known as *indifference* and *weak preference*. These satisfy various properties.

Definition 2.1.3. A relation q on W is:

(i) *symmetric* iff $xqy \Rightarrow yqx$ for any $x, y\varepsilon W$.

(ii) *reflexive* iff xqx for all $x\varepsilon W$.

(iii) *connected* iff xqy or yqx, for any $x, y\varepsilon W$.

(iv) *weakly connected* iff $x \neq y \Rightarrow xqy$ or yqx for $x, y\varepsilon W$.

Definition 2.1.4. For a strict preference relation p, define the symmetric component $I(p)$ called *indifference* by:

$$xI(p)y \text{ iff } \operatorname{not}(xpy) \textit{ and } \operatorname{not}(ypx).$$

Define the reflexive component $R(p)$,called *weak preference,* by.

$$xR(p)y \text{ iff } \; xpy \textit{ or } xI(p)y..$$

Note that since p is assumed irrefexive, then $I(p)$ must be refexive. From the definition $I(p)$ must also be symmetric, although $R\,(p))$ *need* not be. From the definitions:

$$xpy \textit{ or } xI(p)y \textit{ or } ypx,$$

so that $xR(p)y \Leftrightarrow \operatorname{not}(ypx)$. Furthermore either $xR(p)y$ or $yR(p)x$ must be true for any $x, y\varepsilon W$, so that $R(p)$ is *connected.* In terms of an ordinal utility function for p, it is the case that for any x, y in W:

(i) $xI(p)y$ iff $u_p(x) = u_p(y)$

(ii) $xR(p)y$ iff $u_p(x) \geq u_p(y)$

If p is representable by u, then from the natural orderings on the real line, $\mathbb{R}$, it follows that p must satisfy certain consistency properties.

If

$$u(x) > u(y)$$

and

$$u(y) > u(z)$$

it follows that

$$u(x) > u(z)$$

Thus it must be the case that

$$xpy \text{ and } ypz \Rightarrow xpz.$$

This property of a preference relation is known as *transitivity* and may be seen as a desirable property for preference even when p itself is not representable by a utility function. The three consistency properties for preference that we shall use are the following.

Definition 2.1.5. A relation q on W satisfies

(i) *Negative transitivity* iff not(xqy) and not$(yqz) \Rightarrow$ not(xqz)

(ii) *Transitivity* iff xqy and $yqz \Rightarrow xqz$

(iii) *Acyclicity* iff for any finite sequence $x_1, \ldots, x_r$ in W it is the case that if $x_j q x_{j+1}$ for $j = 1, \ldots, r-1$ *then* not$(x_r q x_1)$. If q fails acyclicity then it is called *cyclic*.

The class of strict preference relations on W will be written $B(W)$. If $p \varepsilon B(W)$ and is moreover negatively transitive then it is called a *weak order*. The class of weak orders on W is written $O(W)$. In the same way if p is a transitive strict preference relation then it is called a *strict partial order*, and the class of these is written $T(W)$. Finally the class of *acyclic* strict preference relations on W is written $A(W)$. If $p \varepsilon O(W)$ then it follows from the definition that $R(p)$ is transitive. Indeed $I(p)$ will also be transitive.

Lemma 2.1.1 If $p \varepsilon O(W)$ then $I(p)$ is transitive.

Proof. Suppose $xI(p)y$, $xI(p)z$ but not$(xI(p)z)$. Because of not$(xI(p)z)$ suppose xpz. By asymmetry of p, not(zpx) so $xR(p)z$. By symmetry of $I(p), yI(p)x$ and $zI(p)y$, and thus $yR(p)x$ and $zR(p)y$. But $xR(p)z$ and $zR(p)y$ and $yR(p)x$ contradicts the transitivity of $R(p)$. Hence not(xpz). In the same way not(zpx), and so $xI(p)z$, with the result that $I(p)$ must be transitive. ∎

Lemma 2.1.2. If $p \varepsilon O(W)$ then

$$xR(p)y \text{ and } ypz \Rightarrow xpz.$$

Proof. Suppose $xR(p)y$, ypz and not(xpz).
But

$$\text{not}(xpz) \Leftrightarrow zR(p)x.$$

By transitivity of $R(p)$, $zR(p)y$.

By definition not(ypz) which contradicts ypz by asymmetry. ∎

Lemma 2.1.3. $O(W) \subset T(W) \subset A(W)$.

Proof.

(i) Suppose $p\varepsilon O(W)$ but xpy, ypz yet not(xpz), for some x, y, z. By p asymmetry, not(ypx) and not(zpy). Since $p\varepsilon O(W)$, not(xpz) and not$(zpy) \Rightarrow$ not(xpy). But not(ypx). So $xI(p)y$. But this violates xpy. By contradiction, $p\varepsilon T(W)$.

(ii) Suppose x_jpx_{j+1} for $j = 1, \ldots, r-1$. If $p\varepsilon T(W)$, then x_1px_r. By asymmetry, not(x_rpx_1), so p is acyclic. ∎

2.2. Social Preference Functions

Let the society be $N = \{1, \ldots, i, \ldots, n\}$. A *profile* for N on W is an assignment to each individual i in N of a strict preference relation p_i on W. Such an n-tuple $(p_1, \ldots, p_{n)} will$ be written p. A subset $M \subset N$ is called a *coalition.* The restriction of p to M will be written

$$p_{/M} = (..p_i..\colon i\varepsilon M).$$

If p is a profile for N on W, write xp_Ny iff xp_iy for all xp_iy for all $i\varepsilon N$. In the same way for M a coalition in N write xp_My whenever xp_iy for all $i\varepsilon M$.

Write $B(W)^N$ for the class of profiles on N. When there is no possibility of misunderstanding we shall simply write B^N for $B(W)^N$.

On occasion the analysis concerns profiles each of whose component individual preferences are assumed to belong to some subset $F(W)$ of $B(W)$; for example $F(W)$ might be taken to be $O(W)$, $T(W)$ or $A(W)$. In this case write $F(W)^N$, or F^N, for the class of such profiles.

Let X be the class of all subsets of W. A member $V\varepsilon X$ will be called a *feasible set.*

Suppose that $p\varepsilon B(W)^N$ is a profile for N on W. For some $x, y\varepsilon W$ write "$p_i(x,y)$" for the preference expressed by i on the alternatives x, y under the profile p. Thus "$p_i(x,y)$" will give either xp_iy or $xI(p_i)y$ or yp_ix.

If $f, g\varepsilon B\left(W\right)^N$ are two profiles on W, and $V\varepsilon X$, use "$f_{/M} = g_{/M}$ on V" to mean that for any $x, y\varepsilon V$, any $i\varepsilon M$,

$$\text{“}f_i(x,y)\text{”} = \text{“}g_i(x,y)\text{”}$$

In more abbreviated form write $f^V_{/M} = g^V_{/M}$. Implicitly this implies consideration of a restriction operator

$$\begin{matrix} V \\ M \end{matrix} : B(W)^N \to B(V)^M : f \to f^V_{/M}$$

where $B(V)^M$ means naturally enough the set of profiles for M on V.

A *social preference function* is a method of aggregating preference information, and *only* preference information, on a feasible set to construct a social preference relation.

Definition 2.2.1. A *method of preference aggregation* (MPA), σ, assigns to any feasible set V, and profile p for N on W a strict social preference relation $\sigma(V,p)\varepsilon B(V)$. Such a method is written as $\sigma\colon X \times B^N \to B$. As before write $\sigma(V,p)(x,y)$ for $x, y\varepsilon V$ to mean "the social preference relation declared by $\sigma(V,p)$ between x and y." If $f, g\varepsilon B^N$, write

$$\sigma(V,f) = \sigma(V,g)$$

whenever "$\sigma(V,f)(x,y) = \sigma(V,g)(x,y)$" for any $x, y\varepsilon V$.

Definition 2.2.2. A method of preference aggregation $\sigma\colon X \times B^N \to B$ is said to satisfy the *weak axiom of independence of infeasible alternatives* (II) iff

$$f^V = g^V \Rightarrow \sigma(V, f) = \sigma(V, g).$$

Such a method is called a *social preference function* (SF). Note that an SF, σ, is functionally dependent on the feasible set V. Thus there need be no relationship between $\sigma(V_1, f)$ and $\sigma(V_2, f)$ for $V_2 \subset V_1$ say.

However, suppose $\sigma(V_1, f)$ is the preference relation induced by σ from f on V_1. Let $V_2 \subset V_1$, and let $\sigma(V_1, f)/V_2$ be the preference relation induced by $\sigma(V_1, f)$ on V_2 from the definition

$$[\sigma(V_1, f)/V_2](x, y)\rfloor = “\sigma(V_1, f)(x, y)\rfloor$$

for any $x, y \varepsilon V_2$.

A binary preference function is one consistent with this restriction operator.

Definition 2.2.3. A social preference function σ is said to satisfy the *strong axiom of independence of infeasible alternatives* (II*) iff for $f \varepsilon B(V_1)^N$, $g \varepsilon B(V_2)^N$, and $f^V = g^V$ for $V = V_1 \cap V_2$ non-empty, then

$$\sigma(V_1, f)/V = \sigma(V_2, g)/V.$$

For $\sigma(V_1, f)$ to be meaningful when σ is an SF, we only require that f be a profile defined on V_1. This indicates that II* is an extension property. For suppose f, g are defined on V_1, V_2 respectively, and agree on V. Then it is possible to find a profile p defined on $V_1 \cup V_2$, which agrees with f on V_1 and with g on V_2. Furthermore if σ is an SF which satisfies II*, then

$$\begin{aligned}
\sigma(V_1 \cup V_2, p)/V_1 &= \sigma(V_1, f) \\
\sigma(V_1 \cup V_2, p)/V_2 &= \sigma(V_2, g) \\
\sigma(V_1 \cup V_2, p)/V_1 \cap V_2 &= \sigma(V_1, f)/V = \sigma V_2, g)/V.
\end{aligned}$$

Consequently if $V_2 \subset V_1$ and f is defined on V_1, let f^{V_2} be the restriction of f to V_2. Then for a BF,

$$\sigma(V_2, f^{V_2}) = \sigma(V_1, f)/V_2.$$

The attraction of this axiom is clear. It implies that one can piece together the observed social preferences on various feasible sets to obtain a universal social preference on W.

Moreover in the definition we need only consider V to be a pair of alternatives and construct the social preference $\sigma(f)$ from the pairwise comparisons. It is for this reason that a SF satisfying II* is called a *binary social preference function* (BF). We may regard a BF as a function $\sigma\colon B^N \to B$.

Some care has to be taken in the specification of the domain of an MPA, an SF or a BF. First of all to specify an MPA, the universal set W on which σ is to operate must be defined. Even though $f^V = g^V$ for two profiles f, g and some feasible set V, it need not be the case that $\sigma(V, f) = \sigma(V, g)$.

With regard to an SF, σ, suppose V is some feasible set, and f a profile defined only on V. Then $\sigma(V, f)$ is defined and is a strict preference relation on V. Consequently the domain of σ may be regarded as the *union* of $VxB(V)^N$ across all V in$W$.

Because of the restriction property required of a BF, σ, its domain may be regarded as the union of $B(V)^N$ across all V in W.

To illustrate the differences between an MPA, an SF and a BF, consider the following adaptation of an example due to Plott (1976).

Example 2.2.1. Three individuals i, j, k seek to choose a candidate for a job from a short list $V = \{x, y, z, w\}$. For purposes of illustration take the *universal* set to be

$$W = V \cup \{M\} \cup \{J\} \cup \{S\},$$

where M, J, S stand for Madison, Jefferson and J.S. Mill respectively. The preferences (f) of the individuals are:

i	j	k	Borda Count
M	y	z	z:16
J	z	w	y:15
x	w	x	x:14
y	x	y	M:13
z	M	M	w:12
S	J	J	J:10
w	S	S	S:4

(i) The Borda count is used on W: that is the most preferred candidate if each individual scores 7 and the least preferred 1. On W, z wins with 16, and y is second. Assume social preference on V is induced by restriction from W. With the profile f, we obtain $z\sigma y\sigma x\sigma w$. Now change i's preferences do x is preferred to y to M to J to z to S to w. With this new profile, g, the induced preference on V is $y\sigma zI(\sigma)x\sigma w$, where $zI(\sigma)x$ means z and x are socially indifferent. This decision rule is an MPA, because although $\sigma(V, f) \neq \sigma(V, g)$, f and g are not identical on W. Although the method uses restriction as required for a BF, it satisfies neither II nor II*. This can be seen since

$$f^V = g^V$$

yet

$$\sigma(W, f)/V \neq \sigma(W, g)/V.$$

(ii) Alternatively suppose that on each subset V' of W the Borda count is recomputed. Thus on V, an individual's best alternative scores 4 and the worst 1. The scores for (z, y, x, w) are now $(9, 8, 7, 6)$, so $z\sigma y\sigma x\sigma w$. Clearly σ satisfies II and is an SF, since by definition, if f and g agree on V, so must the scores on V.

However, this method is not a BF, since this social preference cannot be induced by restriction from $\sigma(W, g)$.

More importantly, consider the restriction of the method to binary choice. For example on $\{x, y\}$, x scores 5 and y only 4 so $x\sigma y$. Indeed under this binary majority rule, $z\sigma w\sigma x\sigma y$ yet $y\sigma z$, a *cyclic* preference. Thus the social preference on V cannot be constructed simply by pairwise comparisons.

One method of social decision that is frequently recommended is to assign to each individual i a utility function u_i, representing p_i, and to define the social utility function by

$$u_\sigma(x) = \sum_{i \varepsilon N} \lambda_i u_i(x), \text{ with all } \lambda_i \geq 0.$$

Social preference can be obtained from u_σ in the obvious way by $x\sigma(p)y \Leftrightarrow u_\sigma(x) > u_\sigma(y)$. See for example Harsanyi (1976), Sen (1973), Rawls (1971). Unfortunately in the ordinal framework, each u_i is only defined up to an equivalence relation, and in this setting the above expression has no meaning. In this case $\sigma(p)$ is not well defined. Such a procedure in general cannot be used then to define a social preference function. However, if each feasible set is finite, then as the Borda count example shows we may define $u_i(x) = (v - r_i)$ where $|V| = v$ and r_i is the rank that x has in i's preference schedule. Although this gives a well defined SF, $\sigma(V, p)$, it nonetheless results in a certain inconsistency, since $\sigma(V_1, p_1)$ and $\sigma(V_2, p2)$ may not agree on the intersection $V_1 \cap V_2$, even though p_1 and p_2 do.

Although a BF avoids this difficulty, other inconsistencies are introduced by the strong independence axiom.

2.3. Arrowian Impossibility Theorems

This section considers the question of the existence of a binary social preference function, $\sigma\colon F^N \to F$, where F is some subset of B. In this notation $\sigma\colon F^N \to F$ means the following:

Let V be any feasible set in W, and $F(V)^N$ the set of profiles, defined on V, each of whose component preferences belong to F. The domain of σ is the union of $F(V)^N$ across all V in W. That is for each $f \varepsilon F(V)^N$,we write $\sigma(f)$ for the binary social preference on V, and require that $\sigma(f) \varepsilon F(V)$.

Definition 2.3.1. A BF $\sigma\colon B^N \to B$ satisfies

(i) The *weak Pareto property* (P) iff for any $p \varepsilon B^N$, any $x, y \varepsilon W$,

$$xp_N y \Rightarrow x\sigma(p)y$$

(ii) *Nondictatorship* (ND) iff there is no $i \varepsilon N$ such that for all x, y in W,

$$xp_i y \Rightarrow x\sigma(p)y.$$

A BF σ which satisfies (P) and (ND) and maps $O^N \to O$ is called a *binary welfare function* (BWF).

Arrow's Impossibility Theorem 2.3.1. For N finite, there is no BWF.

This theorem was originally obtained by Arrow (1951). To prove it we introduce the notion of a *decisive coalition.*

Definition 2.3.2. Let M be a coalition, and σ a BF.

(i) Define M to be *decisive under σ for x against y* iff for all $p \varepsilon B^N$

$$xp_M y \Rightarrow x\sigma(p)y.$$

(ii) Define M to be *decisive under σ* iff for all $x, y \varepsilon W, M$ is decisive for x against y.

(iii) Let $\mathbb{D}_\sigma(x, y)$ be the family of decisive coalitions under σ for x against y, and $\mathbb{D}_\sigma$ be the family of decisive coalitions under σ.

To prove the Theorem we first introduce the idea of an ultrafilter.

Definition 2.3.3. A family of coalitions, can satisfy the following properties.

(F1) *monotonicity*: $A \subset B$ and $A \varepsilon \mathbb{D} \Rightarrow B \varepsilon \mathbb{D}$

(F2) *identity*: $N \varepsilon \mathbb{D}$ and $\Phi \not\varepsilon \mathbb{D}$ (where Φ is the empty set)

(F3) *closed intersection*: $A, B\varepsilon\mathbb{D} \Rightarrow A \cap B\varepsilon\mathbb{D}$

(F4) *negation*: for any $A \subset N$, either $A\varepsilon\mathbb{D}$ or $N\backslash A\varepsilon\mathbb{D}$.

A family $\mathbb{D}$ of subsets of N which satisfies (F1), (F2), (F3) is called a *filter*. A filter $\mathbb{D}_1$ is said to be *finer* than a filter $\mathbb{D}_2$ if each member of $\mathbb{D}_2$ belongs to $\mathbb{D}_1$. $\mathbb{D}_1$ is *strictly finer* than $\mathbb{D}_2$ iff $\mathbb{D}_1$ is finer than $\mathbb{D}_2$ and there exists $A\varepsilon\mathbb{D}_1$ with $A\not\varepsilon\mathbb{D}_2$. A filter which has no strictly finer filter is called an *ultrafilter*. A filter is called *free* or *fixed* depending on whether the intersection of all its members is empty or non-empty. In the case that N is finite then by (F2) and (F3) any filter, and thus any ultrafilter, is fixed.

Lemma 2.3.2. (Kirman Sondermann 1972). If $\sigma\colon O^N \rightarrow O$ is a BF and satisfies the weak Pareto property (P), then the family of decisive coalitions, $\mathbb{D}_\sigma$, satisfies (F1), F(2), F(3) and F(4).

We shall prove this lemma below. Arrow's theorem follows from Lemma 2.3.2 since $\mathbb{D}_\sigma$ will be an ultrafilter which defines a unique dictator. This can be shown by the following three lemmas.

Lemma 2.3.3. Let $\mathbb{D}$ be a family of subsets of N, which satisfies (F1), (F2), (F3) and (F4). Then if $A\varepsilon\mathbb{D}$, there is some proper subset B of A which belongs to $\mathbb{D}$.

Proof. Let B be a proper subset of A with $B\not\varepsilon\mathbb{D}$. By (F4), $N\backslash B\varepsilon\mathbb{D}$.

$$A \cap (N\backslash B) = A\backslash B\varepsilon\mathbb{D}.$$

Hence if $B \subset A$, either B or $A\backslash B\varepsilon\mathbb{D}$. ∎

Lemma 2.3.4. If $\mathbb{D}$ satisfies (F1), (F2), (F3) and (F4) then it is an ultrafilter.

Proof. Suppose $\mathbb{D}_1$ is a filter which is strictly finer than $\mathbb{D}$. Then there is some $A, B\varepsilon\mathbb{D}_1$, with $A\varepsilon\mathbb{D}$ but $B\not\varepsilon\mathbb{D}$. By the previous lemma, either $A\backslash B$ or $A \cap \mathbb{B}$ must belong to $\mathbb{D}$. Suppose $A\backslash B\varepsilon\mathbb{D}$. Then $A\backslash B\varepsilon\mathbb{D}_1$. But since $\mathbb{D}_1$ is a filter $(A\backslash B) \cap B = \Phi$ must belong to $\mathbb{D}$, which contradicts (F2). Hence $A \cap B$ belongs to $\mathbb{D}$. But by (F1), $B\varepsilon\mathbb{D}$. Hence $\mathbb{D}$ is an ultrafilter. ∎

Lemma 2.3.5. If N is finite and $\mathbb{D}$ is an ultrafilter, then

$$\bigcap_{\mathbb{D}} A_j = \{i\},$$

where $\{i\}$ is decisive and consists of a single member of N.

Proof. Consider any $A_j\varepsilon\mathbb{D}$, and let $i\varepsilon A_j$. By (F4) either $\{i\}\varepsilon\mathbb{D}$ or $A_j - \{i\}\varepsilon\mathbb{D}$. If $\{i\}\not\varepsilon\mathbb{D}$, then $A_j - \{i\}\varepsilon\mathbb{D}$. Repeat the process a finite number of times to obtain a singleton $\{i\}$, say, belonging to $\mathbb{D}$. ∎

Proof of Theorem 2.3.1. For N finite, by the previous four lemmas, the family of σ-decisive coalitions forms an ultrafilter. The intersection of all decisive coalitions is a single individual i, say. Since this intersection is finite, $\{i\}\varepsilon\mathbb{D}_\sigma$. Thus i is a *dictator*. Consequently any BF $\sigma\colon O^N \rightarrow O$ which satisfies (P) must be dictatorial. Hence there is no BWF. ∎

Note that when N is infinite there can exist a BWF σ (Fishburn, 1970). However, its family of decisive coalitions still form an *ultrafilter*. By the previous lemmas this means that for *any* decisive coalition there is a proper subset which is also decisive. The limit of this process gives what Kirman and Sondermann termed *an invisible dictator*. See Armstrong (1980) and Schmitz (1977) for further discussion on the existence of a BWF when N is an infinite society.

The rest of this section will prove Lemma 2.3.2. The following definitions are required.

Definition 2.3.3. Let σ be a BF, M a coalition, p a profile, $x, y\varepsilon W$.

(i) M is *almost decisive for* x *against* y *with respect to* p iff $xp_My, yp_{N-M}x$ and $x\sigma(p)y$.

(ii) M is *almost decisive for* x *against* y iff for all $p\varepsilon B^N$, xp_My and $p_{N-M} \Rightarrow x\sigma(p)y$. Write $\mathbb{D}^0_\sigma(x,y)$ for the family of coalitions almost decisive for x against y.

(iii) M is *almost decisive* if it is almost decisive for x against y for all $x, y\varepsilon W$. Write $\mathbb{D}^0_\sigma$ for this family.

As before let $\mathbb{D}_\sigma(x,y)$ be the family of coalitions decisive under σ for x against y, and $\mathbb{D}_\sigma$ be the family of decisive coalitions (see Def. 2.3.2). Note that

$$\begin{array}{ccc} \mathbb{D}_\sigma & \subset & \mathbb{D}_\sigma(x,y) \\ \cap & & \cap \\ \mathbb{D}^0_\sigma & \subset & \mathbb{D}^0_\sigma(x,y) \end{array}$$

since being *decisive* is a stronger property than being *almost decisive.*

Lemma 2.3.6. If $\sigma\colon B^N \to B$ is a BF, and M is almost decisive for x against y with respect to some f, then $M\varepsilon\mathbb{D}^0_\sigma(x,y)$.

Proof. Suppose there is some f such that $xf_My, yf_{N-M}x$ and $x\sigma(f)y$. Let g be any profile in B^N which agrees with f on $\{x,y\}$. Since $x\sigma(f)y$ by the strong independence axiom,II*,we obtain $x\sigma(g)y$. So $M\varepsilon\mathbb{D}^0_\sigma(x,y)$. ∎

Lemma 2.3.7. (Sen, 1970). Suppose $\sigma\colon T^N \to T$ is a BF. Then it satisfies

$$\mathbb{D}_\sigma = \mathbb{D}_\sigma(x,y)$$

for any x, y.

Proof.

(i) We seek first to show that

$$\mathbb{D}^0_\sigma(x,y) \subset \mathbb{D}_\sigma(x,z)$$

for any $z \neq x$ or y. Let $M\varepsilon\mathbb{D}^0_\sigma(x,y)$. We need to show that

$$xf_Mz \Rightarrow x\sigma(f)z$$

for any $f\varepsilon T^N$. Let $g\varepsilon T^N$, and suppose xg_Myg_Mz and $yg_{N-M}x$, $yg_{N-M}z$.Thus xg_Mz, by transitivity of g_i, $i\varepsilon M$. Since $M\varepsilon\mathbb{D}^0_\sigma(x,y)$, $x\sigma(g)y$. By (P), $y\sigma(g)z$. By transitivity $x\sigma(g)z$. Let xf_Mz, and choose g such that $f = g$ on $\{x,z\}$. Since $x\sigma(g)z$, by II*, $x\sigma(f)z$, so $M\varepsilon\mathbb{D}_\sigma(x,z)$.

(ii) Now we show that $\mathbb{D}^0_\sigma(x,y) \subset \mathbb{D}_\sigma(z,y)$ for $z \neq x$ or y. Let $M\varepsilon\mathbb{D}^0_\sigma(x,y)$. In the same way, let $h\varepsilon T^N$ with zh_Mxh_My and $zh_{N-M}x, yh_{N-M}x$. Thus zh_My by transitivity of $h_i, i\varepsilon M$. Since $M\varepsilon\mathbb{D}^0_\sigma(x,y)$, $x\sigma(h)y$. By (P), $z\sigma(h)x$. By transitivity $z\sigma(h)y$. Suppose $f\varepsilon T^N$, with $zf_My and$ construct $f = h$ on $\{z,y\}$. By II*, $z\sigma(f)y$, so $M\varepsilon\mathbb{D}_\sigma(z,y)$.

(iii) By reiteration of (i) and (ii), $\mathbb{D}^0_\sigma(x,y) \subset \mathbb{D}_\sigma(u,v)$ for any $u, v\varepsilon W$. But since $\mathbb{D}^0_\sigma(x,y) \subset \mathbb{D}_\sigma(x,y)$, this shows that $\mathbb{D}^0_\sigma(x,y) \subset \mathbb{D}_\sigma$. However, by definition $\mathbb{D}_\sigma \subset \mathbb{D}^0_\sigma(x,y)$, so

$$\mathbb{D}_\sigma = \mathbb{D}^0_\sigma = \mathbb{D}_\sigma(x,y) = \mathbb{D}^0_\sigma(x,y), \text{for any } x, y. \quad ∎$$

Lemma 2.3.8. (Hansson, 1976). If $\sigma\colon T^N \to T$ is a BF and satisfies (P) then $\mathbb{D}_\sigma$ is a *filter*.
Proof.

(F1) Suppose $A \varepsilon \mathbb{D}_\sigma$ and $A \subset B$. Now

$$x f_B y \Rightarrow x f_A y \Rightarrow x\sigma(f)y,$$

so

$$A \varepsilon \mathbb{D}_\sigma(x,y) \Rightarrow B \varepsilon \mathbb{D}_\sigma(x,y)$$

and by the previous lemma, $B \varepsilon \mathbb{D}_\sigma$.

(F2) By (P), $N \varepsilon \mathbb{D}_\sigma$. Suppose that $\Phi \varepsilon \mathbb{D}_\sigma$. But this would imply, for some p, $x p_N y$ and $y\sigma(p)x$ which contradicts (P).

(F3) Suppose that $A, B \varepsilon \mathbb{D}_\sigma$. Let

$$\begin{aligned} V_1 &= A \cap B, \\ V_2 &= A \cap (N \backslash B), \\ V_3 &= (N \backslash A) \cap B, \\ V_4 &= N \backslash (A \cup B). \end{aligned}$$

Define p on $\{x, y, z\}$, in the following way. To each individual in group V_i, for $i = 1, \ldots, 4$, assign the preference p_i in the following fashion:

$$z p_1 x p_1 y$$
$$x p_2 y p_2 z$$
$$y p_3 z p_2 x$$
$$y p_4 x p_4 z.$$

Since

$$\begin{aligned} A &= V_1 \cup V_2 \varepsilon \mathbb{D}_\sigma, \text{ we obtain} x\sigma(p)y, \\ B &= V_1 \cup V_3 \varepsilon \mathbb{D}_\sigma, \text{ we obtain } z\sigma(p)x. \end{aligned}$$

By transitivity, $z\sigma(p)y$. Now $z p_{V_1} y$, $y p_{N-V_1} z$ and $z\sigma(p)y$. By Lemma 2.3.6, $V_1 \varepsilon \mathbb{D}^0_\sigma(z,y) = \mathbb{D}_\sigma$. Thus $A \cap B \varepsilon \mathbb{D}_\sigma$. ∎

This lemma demonstrates that if $\sigma\colon T^N \to T$ is a BF which satisfies (P) then $\mathbb{D}_\sigma$ is a filter. However, if $p \varepsilon O^N$ then $p \varepsilon T^N$, and if $\sigma(p) \varepsilon O$ then $\sigma(p) \varepsilon T$, by Lemma 2.1.3. Hence to complete the proof of Lemma 2.3.2 only the following lemma needs to be shown.

Lemma 2.3.9. If $\sigma\colon O^N \to O$ is a BF and satisfies (P) then $\mathbb{D}_\sigma$ satisfies (F4).

Proof. Suppose $M \not\varepsilon \mathbb{D}_\sigma$. We seek to show that $N \backslash M \varepsilon \mathbb{D}_\sigma$. If for any f, there exist $x, y \varepsilon W$ such that $y f_M x$ and $y\sigma(f)x$, then M would belong to $\mathbb{D}_\sigma(x,y)$ and so be decisive. Thus for any f, there exist $x, y \varepsilon W$, with $y f_M x$ and not $(y\sigma(f)x)$ i.e. $xR(\sigma(f))y$. Now consider $g \varepsilon O^N$, with $g = f$ on $\{x, y\}$ and $x g_{N-M} z, y g_{N-M} z$ and $y g_M z$. By II*, $xR(\sigma(g))y$. By (P), $y\sigma(g)z$. Since $\sigma(g) \varepsilon O$ it is negatively transitive, and by Lemma 2.1.2, $x\sigma(g)z$. Thus $N \backslash M \varepsilon \mathbb{D}_\sigma(x,z)$ and so $N \backslash M \varepsilon \mathbb{D}_\sigma$. ∎

2,4. Power and Rationality

Arrow's theorem showed that there is no binary social preference function which maps weak orders to weak orders and satisfies the Pareto and non-dictatorship requirements when N is finite. Although there may exist a BWF when N is infinite, nonetheless "power" is concentrated in the sense that there is an "invisible dictator." It can be argued that the requirement of negative transitivity is too strong, since this property requires that indifference be transitive. Individual indifference may well display intransitivities, because of just perceptible differences, and so may social indifference. To illustrate the problem with transitivity of indifference, consider the binary social prefence function, called the *weak Pareto rule* written σ_n and defined by:

$$x\sigma_n(p)y \text{ iff } xp_N y.$$

In this case $\{N\} = \mathbb{D}_{\sigma_\ltimes}$. This rule is a BF, satisfies (P) by definition, and is non-dictatorial. However, suppose the preferences are

$$zp_M xp_M y$$

$$yp_{N-M} zp_{N-M} x$$

for some proper subgroup M in N. Since there is not unanimous agreement, this implies $xI(\sigma_n)yI(\sigma_n)z$. If negative transitivity is required, then it must be the case that $xI(\sigma_n)z$. Yet $zp_N x$, so $z\sigma(p)x$. Such an example suggests that the Impossibility Theorem is due to the excessive rationality requirement. For this reason Sen(1970) suggested weakening the rationality requirement.

Definition 2.4.1. A BF $\sigma\colon O^N \to T$ which satisfies (P) and (ND) is called a *binary decision function* (BDF).

Lemma 2.4.1. There exists a BDF.

To show this, say a BF σ satisfies the *strong Pareto property* (P*) iff, for any $p\varepsilon B^N$, $yp_i x$ for no $i\varepsilon N$, and $xp_j y$ for some $j\varepsilon N \Rightarrow x\sigma(p)y$. Note that the strong Pareto property (P*) implies the weak Pareto property (P).

Now define a BF $\overline{\sigma}_n$, called the *strong Pareto rule*, by:

$$x\overline{\sigma}_n(p)y \text{ iff } yp_i x \text{ for no } i\varepsilon N \text{ and } xp_j y \text{ for some } j\varepsilon N.$$

$\overline{\sigma}_n$ may be called the *extension* of σ_n, since it is clear that

$$x\sigma_n\,(p)\,y \Rightarrow x\overline{\sigma}_n(p)y.$$

Obviously $\overline{\sigma}_n$ satisfies (P*) and thus (P). However, just as σ_n violates transitive indifference, so does $\overline{\sigma}_n$. On the other hand $\overline{\sigma}_n$ satisfies transitive strict preference.

Lemma 2.4.2. (Sen, 1970). $\overline{\sigma}_n$ is a BDF.

Proof. Suppose $x\overline{\sigma}_n(p)y$ and $y\overline{\sigma}_n(p)z$. Now

$$x\overline{\sigma}_n(p)y \Leftrightarrow xR(p_i)y \text{ for all } i\varepsilon N \text{ and } xp_j y \text{ for some } j\varepsilon N.$$

Similarly for $\{y, z\}$. By transitivity of $R(p_i)$, we obtain $xR(p_i)z$ for all $i\varepsilon N$. By Lemma 2.1.2, $xp_j z$ for some $j\varepsilon N$. Hence $x\overline{\sigma}_n(p)z$. ∎

While this seems to refute the relevance of the impossibility theorem, note that the only decisive coalition for $\overline{\sigma}_n$ is $\{N\}$. Indeed the strong Pareto rule is somewhat indeterminate, since any individual can effectively veto a decision. Any attempt to make the rule more "determinate" runs into the following problem.

Definition 2.4.2. An *oligarchy* θ_σ for a BF σ is a minimally decisive coalition which belongs to every decisive coalition.

Lemma 2.4.3. (Gibbard, 1969). If N is finite, then any BF $\sigma\colon O^N \to T$ which satisfies P has an oligarchy.

Proof. Restrict σ to $\sigma\colon T^N \to T$. By Lemma 2.3.8, since σ satisfies (P), its decisive coalitions form a filter. Let $\theta = \cap A_j$, where the intersection runs over all $A_j \varepsilon \mathbb{D}_\sigma$. Since N is finite, this intersection is finite, and so $\theta \varepsilon \mathbb{D}_\sigma$. Obviously $\theta - \{i\} \not\varepsilon \mathbb{D}_\sigma$ for any $i \varepsilon \theta$. Consequently θ is a minimally decisive coalition or oligarchy. ▮

The following lemma shows that members of an oligarchy can block social decisions that they oppose.

Lemma 2.4.4. (Schwartz, 1986). If $\sigma\colon O^N \to T$ and $p \varepsilon O^N$, and θ is the oligarchy for σ, then define

$$\begin{aligned} \theta_x(p) &= \{i \varepsilon \theta \colon x p_i y\} \\ \theta_y(p) &= \{j \varepsilon \theta \colon y p_j x\}. \end{aligned}$$

Then

(i) $\theta_x(p) \neq \Phi$ and $\theta = \theta_x(p) \cup \theta_y(p) \Rightarrow$ not $(y\sigma(p)x)$

(ii) $\theta_x(p) \neq \Phi, \theta_y(p) \neq \Phi$ and $\theta = \theta_x \cup \theta_y \Rightarrow xI(\sigma(p))y$.

Individuals in the oligarchy may thus block decisions in the sense implied by this lemma. From these results it is clear that BDF must concentrate power within some group in the society. If the oligarchy is large, as for $\overline{\sigma}_n$, then we may infer that decision making costs would be high. If the oligarchy is small, then one would be inclined to reject the rule on normative qrounds. Consider for a moment an economy where trades are permitted between actors. With unrestricted exchange any particular coalition M is presumably decisive for *certain* advantageous trades. If we require the resulting social preference to be a BDF, then by Lemma 2.3.7, this coalition M has to be (globally) decisive. Consequently there *must* be some oligarchy. In a free exchange economy there is however no oligarchy, and so the social preference relation must violate either the fundamental independence axiom II*, or the rationality condition. This would seem to be a major contradiction between social choice theory and economic equilibrium theory.

Lemma 2.4.3 suggests that the rationality condition be weakened even further to acyclicity. It will be shown below that acyclicity of a BF is sufficient to define a well behaved choice procedure.

Definition 2.4.3. A BF $\sigma\colon A^N \to A$ which satisfied (P) is called a *binary acyclic preference function* (BAF).

Definition 2.4.4. Let $\mathbb{D} = M_1, \ldots, M_r$ be a family of subsets of N. $\mathbb{D}$ is called a *prefilter* iff $\mathbb{D}$ satisfies (F1), (F2) and the following:

(F0) *non-empty intersection*: the intersection $\kappa(\mathbb{D}) = M_1 \cap M_2 \cdots \cap M_r$ is non-empty.

The set $\kappa(\mathbb{D})$ is called the *collegium* of $\mathbb{D}$.

If σ is a BF, $\mathbb{D}_\sigma$ is its family of decisive coalitions, and $\kappa_\sigma = \kappa(\mathbb{D}_\sigma)$ is non-empty, then σ is said to be *collegial.* Otherwise σ is said to be *non-collegial.*

Theorem 2.4.5. (Brown, 1973). If $\sigma\colon A^N \to A$ is a BF which satisfies (P) then σ is collegial and $\mathbb{D}_\sigma$ is a prefilter.

Proof. (F1) and (F2) follow as in Lemma 2.3.8. To prove (F0), suppose there exists $\{M_j\}_{j=1}^r$ where each $M_j \varepsilon \mathbb{D}_{\sigma'}$ yet this family has empty intersection. Let

$$V = \{a_1, \ldots, a_r\}$$

be a collection of distinct alternatives. For each pair $\{a_j, a_{j+1}\}$, $j = 1, \ldots, r-1$, let p^j be a profile defined on $\{a_j, a_{j+1}\}$ such that $a_j p_i^j a_{j+1}$ for all $i \varepsilon A_j$. Thus $a_j \sigma(p^j) a_{j+1}$. In the same way let p^r be defined on $\{a_r, a_1\}$ such that $a_r p_i^r a_1$ for all $i \varepsilon A_r$. Thus $a_r \sigma(p^r) a_1$. Now extend $\{p^1, \ldots, p^r\}$ to a profile p on V, in such a way that each p_i is acyclic.[5] By the extension property of II*,

$$a_j \sigma(p^j) a_{j+1} \Leftrightarrow a_j \sigma(p) a_{j+1} \text{ etc.}$$

Hence

$$a_1 \sigma(p) a_2 \sigma(p) a_3 \cdots a_r \sigma(p) a_1$$

This gives an acyclic profile p such that $\sigma(p)$ is cyclic. By contradiction the family $\{M_j\}_{j=1}^r$ must have non-empty intersection. ▮

Even acyclicity requires some concentration in power, though the existence of a collegium is of course much less unattractive than the existence of an oligarchy or dictator.

The next section turns to the question of the existence of choice procedures associated with binary preference functions, and relates consistency properties of these procedures to rationality properties of the preference functions.

2.5. Choice Functions

Instead of seeking a preference function σ that satisfies certain rationality conditions, one may seek a procedure which "selects" from a set V some subset of V, in a way which is determined by the profile.

Definition 2.5.1. A *choice function* C is a mapping $C\colon X \times B^N \to X$ with the property that $\Phi \neq C(V,p) \subset V$ for any $V \varepsilon X$. Note that the notational convention that is used *only* requires that the profile p be defined on V. If f is defined on V_1, g is defined on V_2 and $f^V = g^V$ for $V = V_1 \cap V_2 \neq \Phi$, then it must be the case that $C(V, f^V) = C(V, g^V)$. Thus by definition a choice function satisfies the analogue of the weak independence axiom (II). Note that there has as yet been no requirement that C satisfy the analogue of the strong independence axiom.

Definition 2.5.2. A choice function $C\colon XxB^N \to X$ satisfies the *weak axiom of revealed preference* (WARP) iff wherever $V \subset V'$, and p is defined on V', with $V \cap C(V',p) \neq \Phi$, then

$$V \cap C(V',p) = C(V, p^V),$$

where p^V is the restriction of p to V. Note the analogue with (II*). If we write $C(V',p)/V$ for $V \cap C(V',p)$ when this is non-empty, then WARP requires that

[5] A later result, Lemma 3.2.6, shows that this can indeed be done, as long as r is of sufficient cardinality.)

$$C(V, p^V) = C(V', p)/V.$$

Definition 2.5.3.

(i) A choice function $C\colon X \times B^N \to X$ is said to be *rationalized* by an SF $\sigma\colon X \times B^N \to B$ iff for any $V \varepsilon X$ and any $p \varepsilon B^N$,

$$C(V, p) = \{x\colon y\sigma(V, p)x \text{ for no } y\varepsilon V\}.$$

(ii) A choice function $C\colon X \times B^N \to X$ is said to be *rationalized* by a BF $\sigma\colon B^N \to B$ iff for any $p\varepsilon B^N$, and any $x, y\varepsilon W, x \neq y$:

$$C(\{x, y\}, p) = \{x\} \Leftrightarrow x\sigma(p)y.$$

(iii) A choice function $C\colon X \times B^N \to X$ is said to satisfy the *binary choice axiom* (BICH) iff there is a BF $\sigma\colon B^N \to B$ such that for any $V\varepsilon X$, any $p\varepsilon B^N$,

$$C(V, p) = \{x\varepsilon V\colon y\sigma(p)x \text{ for no } y\varepsilon V\}.$$

Say C satisfies BICH w.r.t. σ in this case.

(iv) Given a choice function $C\colon X \times B^N \to X$ define the induced BF $\sigma_C\colon B^N \to B$ by

$$C(\{x, y\}, p) = \{x\} \Leftrightarrow x\sigma_C(p)y.$$

(v) Given a BF $\sigma\colon B^N \to B$ define the *choice procedure* $C_\sigma\colon XxB^N \to X$ by

$$C_\sigma(V, p) = \{x\varepsilon V\colon y\sigma(p)x \text{ for no } y\varepsilon V\}$$

Note that $C_\sigma(V, p)$ may be empty for some V, p.

Lemma 2.5.1. If C satisfies BICH w.r.t. σ then σ rationalizes C.
Proof.

(i) $C(\{x, y\}, p) = \{x\} \Rightarrow$not $(y\sigma(p)x$. If $yI(\sigma(p))x$ then not $(x\sigma(p)y)$, so $y\varepsilon C(\{x, y\}, p)$. Hence $C(\{x, y\}, p) = \{x\} \Rightarrow x\sigma(p)y$.

(ii) $x\sigma(p)y \Rightarrow$ not $(y\sigma(p)x)$. Hence $C(\{x, y\}, p) = \{x\}$. ∎

Another way of putting this lemma is that if $C = C_\sigma$ is a choice function then $\sigma = \sigma_C$.
In the following we delete reference to p when there is no ambiguity, and simply regard C as a mapping from X to itself.
Example 2.5.1.

(i) Suppose C is defined on the pair sets of $W = \{x, y, z\}$ by

$$\begin{aligned} C(\{x,y\}) &= \{x\}, \\ C(\{y,z\}) &= \{y\} \end{aligned}$$

and

$$C(\{x,z\}) = \{z\}.$$

If C satisfies BICH w.r.t. σ, then it is necessary that $x\sigma y\sigma z\sigma x$, so

$C(\{x, y, z\}) = \Phi$. Hence C cannot satisfy BICH.

(ii) Suppose

$$\begin{aligned} C(\{x,y\}) &= \{x\} \\ C(\{y,z\}) &= \{y,z\}. \end{aligned}$$

If

$$C(\{x,z\}) = \{x,z\},$$

then $x\sigma yI(\sigma)z$ and $xI(\sigma)z$, so

$$C(\{x,y,z\}) = \{x,z\}.$$

While C satisfies BICH w.r.t. σ, σ does not give a weak order, although σ may give a strict partial order.

Theorem 2.5.2. (Sen, 1970). Let the universal set, W, be of finite cardinality.

(i) If a choice function C satisfies BICH w.r.t. σ, then $\sigma = \sigma_C$ is a BAF.

(ii) If σ is a BAF then C_σ, restricted to $X \times A^N$, is a choice function.

Proof.

(i) By Lemma 2.5.1, if C satisfies BICH w.r.t. σ then σ rationalizes C, and so by definition the induced BF, σ_C, is identical to σ. We seek to show that σ is a BAF, or that $\sigma\colon A^N \to A$. Suppose on the contrary that σ is not a BAF. Since σ is a BF and W is of finite cardinality, this assumption is equivalent to the existence of a finite subset $V = \{a_1, \ldots, a_r\}$ of W, a profile $p\varepsilon A(V)^N$, and a cycle

$$a_1\sigma(p)a_2\sigma(p)\cdots a_r\sigma(p)a_1.$$

Let $a_r \equiv a_0$. Then for each $a_j\varepsilon V$ it is the case that $a_{j-1}\sigma(p)\sigma_j$. Since C satisfies BICH with respect to σ, it is evident that $C(V,p) = \Phi$. By contradiction σ is a BAF.

(ii) We seek now to show that for any finite set V, if $p\varepsilon A(V)^N$ and $\sigma(p)\varepsilon A(V)$ then $C_\sigma(V,p) \neq \Phi$. Let $I(\sigma(p))$ and $R(\sigma(p))$ represent the indifference and weak preference relations defined by $\sigma(p)$. Suppose that $V = \{x_1, \ldots, x_r\}$. If $x_1I(\sigma(p))x_2\ldots x_{r-1}I(\sigma(p))x_r$ then $C_\sigma(V,p) = V$. So suppose that for some $a_1, a_2\varepsilon V$ it is the case that $a_2\sigma(p)a_1$. If $a_2\not\varepsilon C_\sigma(V,p)$ then there exists a_3, say, such $a_3\sigma(p)a_2$. If $a_1\sigma(p)a_3$, then by acyclicity, not $(a_2\sigma(p)a_1)$. Since $\sigma(p)$ is a strict preference relation, this is a contradiction. Hence not $(a_1\sigma(p)a_3)$, and so $a_3\varepsilon C_\sigma(\{a_1,a_2,a_3\},p)$. By induction, $C_\sigma(V',p) \neq \Phi \Rightarrow C_\sigma(V'',p) \neq \Phi$ whenever $|V'|+1 = |V''|$ and $V' \subset V'' \subset W$. Thus $C(V,p) \neq \Phi$ for any finite subset V of W. ∎

Lemma 2.5.3. (Schwartz, 1976). A choice function C satisfies BICH iff for any V_1, V_2,

$$C(V_1) \cap C(V_2) = C(V_1 \cup V_2) \cap V_1 \cap V_2.$$

Consider for the moment $V_1 \subset V_2$. By the above

$$C(V_1 \cup V_2) \cap V_1 \cap V_2 = C(V_2)V_1 = C(V_1) \cap C(V_2) \subseteq C(V_1)$$

Brown (1973) had shown this earlier. Since this is part of the WARP condition, WARP must imply BICH.

Lemma 2.5.4. (Arrow, 1959). A choice function, C, satisfies WARP iff C satisfies BICH and σ_C is a BF $\sigma_C \colon O^N \to O$.

Even though WARP is an attractive property of a choice function, it requires that σ_C satisfy a strong rationality condition sufficient to induce a dictator. Consider now the properties of a choice function when $\sigma_C \colon T^N \to T$.

Definition 2.5.4. The choice function $C \colon X \to X$ satisfies

(i) *independence of path* (IIP) iff

$$C(\cup_{j=1}^{r} C(V_j)) = C(V)$$

whenever

$$V + \cup_{j=1}^{r} V_j.$$

(ii) *exclusion* (EX) iff

$$V_1 \subset V \backslash C(V) \Rightarrow C(V \backslash V_1) \subseteq C(V).$$

Example 2.5.2. To illustrate (EX), consider Example 2.2.1 above and let C be the procedure which selects from V the top-most ranked alternative under the Borda count on V. Thus suppose $V = \{x, y, z, w\}$ and consider the profile

i	j	k
x	y	z
y	z	w
z	w	x
w	x	y

On V, the Borda count for $\{z, y, x, w\}$ is $\{9, 8, 7, 6\}$. Thus $C(V) = \{z\}$. Let $V_1 = \{w\}$, and observe that $V_1 \subset V \backslash C(V)$. Now perform the Borda count on $V \backslash V_1 = \{x, y, z\}$. However, $C(\{x, y, z\}) = \{x, y, z\} \not\subset \{z\}$, so EX is violated. The exclusion axiom is sometimes confused with the independence of infeasible alternatives for choice functions. Schwartz, (1976) and Plott (1970, 1973) have examined the nature of the conditions (EX) and IIP..

Lemma 2.5.5. (Schwartz, 1976). A choice function C satisfies (EX) iff C satisfies BICH *and* σ_C is a BF : $T^N \to T$.

Lemma 2.5.6. (Plott, 1970, 1973).

(i) If a choice function C satisfies II P then the BF $\sigma_C \colon \to T^N$ to T and $C \subset C_{\sigma_C}$.

(ii) If $\sigma \colon T^N \to T$ is a BF, then C_σ satisfies IIP.

Note that if a choice function C satisfies II P then $x\varepsilon C(V)$ implies there is no y st $y\sigma_C x$. Suppose if the following property on C is satisfied: [for all $x, y, \varepsilon V, C\{x, y\}) = \{x, y\} \Rightarrow C(V) = V$]. Then if C satisfies II P it is the case that $C = C_{\sigma_C}$. Since non-oligarchic binary preference functions cannot map $T^N \rightarrow T$, Ferejohn and Grether (1977) have proposed weakening II P in the following way.

Definition 2.5.5. A choice function $C\colon X \rightarrow X$ satisfies *weak path independence* (*II P) iff

$$C(\cup_{j=1}^r C(V_j)) \subset C(V)$$

whenever

$$V = \cup_{j=1}^r V.$$

Lemma 2.5.7. (Ferejohn, Grether, 1977). Let C be a choice function $C\colon X \times B^N \rightarrow X$ which satisfies *II P. If V is a $\sigma_C(p)$ cycle, then

(i) $C(V, p) = V$.

(ii) Moreover, if for any $x\varepsilon W \backslash V$ there is some set $Y \subset W$ such that

$$C(Y \cup \{x\}, p) \subset V,$$

then

$$V \subset C(W, p).$$

Example 2.5.3. If majority rule is used with the profile given in Example 2.5.2, then there is a cycle

$$z\sigma(p)w\sigma(p)x\sigma(p)y\sigma(p)z.$$

So any choice function C which satisfies *II P has to choose $C(V, p) = V = \{x, y, z, w\}$. However, $zp_N w$, so the choice function can choose alternatives which are beaten under the weak Pareto rule (i.e., are not Pareto optimal).

If one seeks a choice function which satisfies the strong consistency properties of WARP or EX, then choices must be made by binary comparisons (BICH), and consequently the Arrowian Impossibility theorems are relevant. If one seek only II P, then $C \subset C_{\sigma_C}$, and again binary comparisons must be made, so the Impossibility theorems are once more relevant. The attraction of *II P is that it permits choice to be done by division. Suppose a decision problem, V, is divided into components V_j, choice made from V_j, and then choice made from these. Then the resultant decision must be compatible with whatever choice would have been made from V.* II P would seem to be a minimal consistency property of a choice procedure. Unfortunately it requires the selection of cycles, no matter how large these be.

The next section examines the occurrence of cycles under general voting rules. Since cycles, and particular non-Paretian cycles, generally occur under such rules, there is a contradiction between implementability (or path independence) and Pareto optimality for general voting processes.

3. Voting Rules

3.1. Simple Binary Preferences Functions

The previous section showed that for a binary preference function σ to satisfy certain rationality postulates it is *necessary* that the family of decisive coalitions obey various filter properties. A natural question is whether the previous restrictions on power, imposed by the filter properties, are *sufficient* to ensure rationality. In general, however, this is not the case. To see this, for a given class of coalitions define a new BF as follows.

Definition 3.1.1. Let N be a fixed set of individuals, and $\mathbb{D}$ a family of subsets of N. Define the BF $\sigma^{\mathbb{D}}\colon B^N \to B$ by:

$$x\sigma^{\mathbb{D}}(p)y \Leftrightarrow \{i\varepsilon N\colon xp_iy\}\varepsilon\mathbb{D}, \text{ whenever } x, y\varepsilon W.$$

For a given BF $\sigma\colon B^N \to B$, $\mathbb{D}_\sigma$ is defined to be its family of decisive coalitions. Consequently there are two transformations:

$$\sigma \to \mathbb{D}_\sigma \text{ and } \mathbb{D}_\sigma \to \sigma^{\mathbb{D}_\sigma}.$$

In terms of these transformations, the previous results may be written:

Lemma 3.1.1. If σ is a BF which satisfies (P) and

(i) $\sigma\colon O^N \to O$ then $\mathbb{D}_\sigma$ is an ultrafilter

(ii) $\sigma\colon T^N \to T$ then $\mathbb{D}_\sigma$ is a filter

(iii) $\sigma\colon A^N \to A$ then $\mathbb{D}_\sigma$ is a prefilter.

Lemma 3.1.2. (Ferejohn, 1977). If $\mathbb{D}$ is

(i) an ultrafilter then $\sigma^{\mathbb{D}}\colon O^N \to O$ is a BF and satisfies (P)

(ii) a filter then $\sigma^{\mathbb{D}}\colon T^N \to T$ is a BF and satisfies (P)

(iii) a prefilter then $\sigma^{\mathbb{D}}\colon A^N \to A$ is a BF and satisfies (P).

However, even though $\mathbb{D}_\sigma$ satisfies one of the filter properties, σ need not satisfy the appropriate rationality property. The problem is that the transformation

$$\sigma \to \sigma^{\mathbb{D}_\sigma}$$

is "structure forgetting." It is easy to see that for any $x, y\varepsilon W, p\varepsilon B^N$,

$$x\sigma^{\mathbb{D}_\sigma}(p)y \Rightarrow x\sigma(p)y$$

Thus $\sigma^{\mathbb{D}_\sigma} \subset \sigma$. For this reason it may be the case that, for some $x, y, p\colon x\sigma(p)y$ although not $(x\sigma^{\mathbb{D}_\sigma}(p)y$. To see this consider the following example due to Ferejohn and Fishburn (1979).

Example 3.1.1. Let $N = \{1, 2\}, W = \{x, y, z\}$ and T be the cyclic relation

$$xTy, yTz, zTx$$

Define

$$x\sigma(p)y \Leftrightarrow xp_1y \text{ or } [xI(p_1)y \text{ and } xTy].$$

It follows from this definition that

$$\mathbb{D}_\sigma = \{\{1\}, \{1, 2\}\} \text{ is an ultrafilter.}$$

Obviously $x\sigma^{\mathbb{D}_\sigma}(p)y \Leftrightarrow xp_1y$. Hence $\sigma^{\mathbb{D}_\sigma}\colon O^N \to O$ is dictatorial. On the other hand if p is a profile under which $\{1\}$ is indifferent on $\{x, y, z\}$ then $x\sigma(p)y\sigma(p)z\sigma(p)x$, so $\sigma(p)$ is cyclic.

Definition 3.1.2. If f σ_1, σ_2 are two binary preference functions on W, and for all $p \varepsilon B^N$

$$x\sigma_1(p)y \Rightarrow x\sigma_2(p)y \text{ for any } x, y \varepsilon W$$

then say that σ_2 is *finer* than σ_1, and write $\sigma_1 \subseteq \sigma_2$ If in addition $x\sigma_2(p)y$ yet not $(x\sigma_1(p)y)$ for some x, y, then say σ_2 is *strictly finer* than σ_1, and write $\sigma_1 \subset \sigma_2$.

If σ_2 is strictly finer than σ_1 then σ_1 may satisfy certain rationality properties, such as acyclicity, although σ_2 need not. On the other hand if σ_1 fails a rationality properly, such acyclicity, then so will σ_2.

From the above discussion, $\sigma^{\mathbb{D}_\sigma} \subseteq \sigma$. Indeed, in Example 3.1.1, σ is strictly finer than $\sigma^{\mathbb{D}_\sigma}$, and is cyclic, even though .$\sigma^{\mathbb{D}_\sigma}$ is acyclic. To induce rationality conditions on σ from properties of $\mathbb{D}_{\sigma'}$ we can require $\sigma = \sigma^{\mathbb{D}_\sigma}$ by assuming certain additional properties on σ.

Definition 3.1.3. Let p, q be any profiles in B^N and x, y alternatives in W. A BF σ is

(i) *decisive* iff $\{i \varepsilon N : xp_iy\} = \{i \varepsilon N : xq_iy\}$ implies that $x\sigma(p)y \Rightarrow x\sigma(q)y$.

(ii) *neutral* iff

$$\{i : xp_iy\} = \{i : aq_ib\}$$

and

$$\{j : yp_jx\} = \{j : bq_ja\}$$

implies that $\sigma(p)(x, y) = \sigma(q)(a, b)$.

(iii) *monotonic* iff

$$\{i : xp_iy\} \subset \{iaq_ib\}$$

and

$$\{j : yp_jx\} \supset \{j : bq_ja\}$$

implies that $[x\sigma(p)y \Rightarrow a\sigma(q)b]$.

(iv) *anonymous* iff

$$\sigma(p) = \sigma(s(p))$$

where $s : N \to N$ is any permutation of N, and

$$s(p) = (p_{s(1)}, p_{s(2)}, \ldots, p_{s(n)}).$$

(v) *simple* iff σ is *decisive* and *neutral.*

Note that monotonicity implies neutrality and that neutrality implies the strong independence axion (II*).

To distinguish between neutrality and decisiveness consider the following example, adapted from Ferejohn and Fishburn (1979).

Example 3.1.2.

(i) Let $\sigma = \sigma_n \cup \sigma'$ on $W = \{a, b, c\}$, where, as before,

$$x\sigma_n(p)y \Leftrightarrow xp_Ny,$$

and for the fixed pair $\{a, b\}$,

$$a\sigma'(p)b \text{ iff } ap_1b \text{ and } aI(p_i)b, \forall i \neq 1.$$

It is clear that $\mathbb{D}_\sigma = \{N\}$. However, σ is *not* decisive. To see this construct two profiles p, q such that:

$$aq_1b \text{ yet } bq_ia \text{ for } i \neq 1$$

$$ap_1b \text{ and } aI(p_i)b \text{ fo r } i \neq 1$$

$$\text{with } p = q \text{ on } \{a, b\}.$$

Although $\{i \colon aq_ib\} = \{i \colon ap_ib\}$ it is the case that $a\sigma(p)b$ yet not$[a\sigma(q)b]$. Hence σ is neither decisive nor neutral.

(ii) Let $\sigma = \sigma_n \cup \sigma'$ where for any $x, y \varepsilon W$ $x\sigma'(p)y$ iff xp_1y and $xI(p_i), \forall i \neq 1$.

As above, σ is not decisive, but it is neutral.

(iii) Let $\sigma = \sigma_n \cup \sigma'$ where σ' is the decisive BF defined by

$$\mathbb{D}_{\sigma'}(a, b) = \{1\}, \mathbb{D}_{\sigma'}(b, c) = \{2\}, \mathbb{D}_{\sigma'}(c, a) = \{3\}.$$

While σ is decisive, it is not neutral.

Lemma 3.1.3. A BF σ is *simple* iff $\sigma = \sigma^{\mathbb{D}_\sigma}$.

Proof. ($\Rightarrow$): Suppose that σ is decisive and neutral.

It is evident that $\sigma^{\mathbb{D}_\sigma} \subset \sigma$. So suppose $x\sigma(p)y$, for some x, y, p. We need to show that

$$\{i \varepsilon N \colon xp_iy\} = M \text{ belongs to } \mathbb{D}_\sigma.$$

For any pair $a, b \varepsilon W$, construct a profile q by

$$p_i(x, y) = q_i(a, b) \colon i \varepsilon N.$$

so preferences on $\{a, b\}$ are identical to $\{x, y\}$. By neutrality,

$$\sigma(p)(x, y) = \sigma(q)(a, b).$$

Since $x\sigma(p)y$, we obtain $a\sigma(q)b$. By decisiveness, for any profile q with aq_ib, for $i \varepsilon M$, then $a\sigma(q)b$.Thus M belongs to $\mathbb{D}_\sigma$,and $\sigma \subset \sigma^{\mathbb{D}_\sigma}$. Hence $\sigma^{\mathbb{D}_\sigma} = \sigma$.

($\Leftarrow$)Since $\mathbb{D}_\sigma$ is a fixed class of coalitions, decisiveness and neutrality of $\sigma^{\mathbb{D}_\sigma}$ and thus σ follow immediately. ∎

From now on call a simple BF a *voting rule*. To illustrate various kinds of voting rules consider the following.

Definition 3.1.4.

(i) A voting rule, σ, is called a *simple weighted majority rule* iff:

(a) each individual i in N is assigned a real valued integer *weight* $w(i) \geq 0$;

(b) each coalition M is assigned the weight

$$w(M) = \sum_{i \varepsilon M} w(i);$$

(c) q is a real valued integer with $\frac{w(N)}{2} < q \leq w(N)$ such that $M \varepsilon \mathbb{D}_\sigma$ iff $w(M) \geq q$;

(d) $\sigma = \sigma^{\mathbb{D}_\sigma}$.

(ii) This voting rule is written $\sigma_{q(w)}$, where

$$q(w) = [q\colon w(1), \ldots, w(i), \ldots, w(n)].$$

(iii) If

$$q(w) = [q\colon 1, \ldots, 1, \ldots, 1].$$

where $w(i) = 1$ for each $i \varepsilon N$, and $q > \frac{n}{2}$.then the voting rule is called the *simple q-majority rule*, or q-rule, and denoted σ_q

(iv) *Simple majority rule*, written as σ_m, is the q-rule defined in the following way:

$$\begin{aligned} \text{if } n &= 2k+1 \text{ is odd, then } q = k+1 = m; \\ \text{if } n &= 2k \text{ is even, then } q = k+1 = m. \end{aligned}$$

In the case $q = n$, we obtain the *weak Pareto rule* σ_n, mentioned in §2.4. Note that σ_q is *anonymous* as well as *simple*.

For conveneience we shall often refer to a simple weighted majority rule as a $q(w) - rule$.

Two further properties of a voting rule σ are as follows.

Definition 3.1.5. A voting rule σ is

(i) *proper* iff for any $A, B \varepsilon \mathbb{D}_\sigma, A \cap B \neq \Phi$

(ii) *strong* iff $A \not\varepsilon \mathbb{D}_\sigma$ then $N \backslash A \varepsilon \mathbb{D}_\sigma$.

For example consider a $q(w)$-simple weighted majority rule, σ. Because $q > \frac{w(N)}{2}$ then $M \varepsilon \mathbb{D}_\sigma$ implies that

$$w(N \backslash A) = w(N) \backslash w(A) < \frac{w(N)}{2}.$$

Hence, if $B \subset N \backslash A$ then $B \not\varepsilon \mathbb{D}_\sigma$. Thus σ must be proper. On the other hand suppose σ is simple majority rule with $|N| = 2k$, an even integer. Then if $|A| = k$, $A \not\varepsilon \mathbb{D}_{\sigma'}$ but $|N \backslash A| = k$ and $N \backslash A \not\varepsilon \mathbb{D}_\sigma$. Thus σ is not strong.However, if $|N| = 2k+1$, an odd integer, and $|A| = k$ then $A \not\varepsilon \mathbb{D}_\sigma$ but $|N \backslash A| = k+1$ and so $N \backslash A \varepsilon \mathbb{D}_\sigma$. Thus σ is strong. Another interpretation of these terms is as follows. If $A \not\varepsilon \mathbb{D}_\sigma$ then A is said to be *losing*. On the other hand if A is such that $N \backslash A \not\varepsilon \mathbb{D}_\sigma$ then call A *blocking*. If σ is strong, then no losing coalition is blocking, and if σ is proper then every winning coalition is blocking.

Given a $q(w)$-rule, σ, it is possible to define a new rule $\bar{\sigma}$, called the *extension* of σ, such that $\bar{\sigma}$ is finer than σ.

Definition 3.1.6.

(i) For a $q(w)$ rule, σ, define its extension $\bar{\sigma}$ by

$$x\bar{\sigma}(p)y \Leftrightarrow w(M_{xy}) \geq q\left(\frac{w(M_{xy}) + w(M_{xy})}{w(N)}\right)$$

where

$$M_{xy} = \{i\colon xp_i y\}$$

and

$$M_{yx} = \{j\colon yp_j x\}.$$

Write $\bar{\sigma}_q$ for the extension of the simple q-majority rule, σ_q. Then

$$x\bar{\sigma}_q(p)y \Leftrightarrow |M_{xy}| \geq q\left(\frac{|M_{xy}| + |M_{yx}|}{|N|}\right).$$

(ii) The *weak Pareto rule*, σ_n, is defined by

$$x\bar{\sigma}_n(p)y \Leftrightarrow |M_{xy}| \geq n.$$

(iii) The *strong Pareto rule*, $\bar{\sigma}_{n'}$, is defined analogously by

$$x\bar{\sigma}_n(p)y \Leftrightarrow |M_{xy}| \geq |M_{xy}| + |M_{yx}|.$$

That is to say

$$|M_{yx}| = \Phi.$$

(iv) *Plurality rule,* written σ_{plur}, is defined by:

$$x\sigma_{plur}(p)y \Leftrightarrow |M_{xy}| > |M_{yx}|.$$

Lemma 3.1.4. The simple q-majority rules and their extensions are *nested*: i.e., for any $q, n/2 < q \leq n$,

$$\begin{array}{ccc} \sigma_n & \subset & \bar{\sigma}_n \\ \cap & & \cap \\ \sigma_q & \subset & \bar{\sigma}_q \\ \cap & & \cap \\ \sigma_m & \subset & \bar{\sigma}_m \end{array}$$

where as before $\sigma_1 \subset \sigma_2$ iff $x\sigma_1(p)y \Rightarrow x\sigma_2(p)y$ wherever $x, y\varepsilon W, p\varepsilon B^N$.

Notice that it is possible that $x\sigma_{plur}(p)y$ yet not$[x\sigma_m(p)y]$. For example, if $n = 4$, and $|M_{xy}| = 2$, $|M_{yx}| = 1$, we obtain $x\sigma_{plur}(p)y$. However,

$$|M_{xy}| < \frac{3}{4}[|M_{xy}| + |M_{yx}|]$$

so not$[x\bar{\sigma}_m(p)y]$.

From Brown's result (Theorem 2.4.5) for a voting rule, σ, to be *acyclic* it is mecessary that there be a collegium κ. Indeed, for a collegial voting rule, each member i of the collegium has the veto power:

$$xp_i y \Rightarrow \quad \text{not } (y\sigma(p)x).$$

As we have seen $\bar{\sigma}_n$ maps O^N to T, and obviously enough σ_n maps $T^N \to T$.

However, any anonymous q-rule σ, with $q < n$, is non-collegial, and so it is possible to find a profile p such that $\sigma(p)$ is *cyclic*.

The next section shows that such a profile must be defined on a feasible set containing sufficiently large a number of alternatives.

3.2. Acyclic Voting Rules on Restricted Sets of Alternatives

In this section we shall show that when the cardinality of the set of alternatives is suitablly restricted, then a voting rule will be acyclic.

Let $B(r)^N$ be the class of profiles, each defined on a feasible set of *at most* r alternatives, and let $F(r)^N$ be the natural restriction to a subclass defined by $F \subset B$. Thus $A(r)^N$ is the set of *acyclic* profiles defined on feasible sets of cardinality at most r.

Lemma 3.2.1. (Ferejohn and Grether, 1974). Let $q^r = \left(\frac{r-1}{r}\right) n$, for a given $|N| = n$.

(i) A q-rule maps $A(r)^N \to A(r)$ iff $q > q^r$.

(ii) The extension of a q-rule maps $O(r)^N \to A(r)$ iff $q > q^r$.

Comment 3.2.1. Note that the inequality $q > \left(\frac{r-1}{r}\right) n$ can be written $rq > rn - n$ or $r < \frac{n}{n-q}$ if $q \neq n$. In the case $q \neq n$, if we define the integer $v(n,q)$ to be the greatest integer which is strictly less then $\frac{q}{n-q}$, then the inequality $q > q^r$ can be written $r \leq v(n,q) + 1$.

Lemma 3.2.1. can be extended to cover the case of a general non-collegial voting rule where the restriction on the size of the alternative set involves, not$v(n,q)$, but the *Nakamura* number of the rule.

Definition 3.2.1.

(i) Let $\mathbb{D}$ be a family of subsets of N. If the collegium, $\kappa(\mathbb{D})$, is non-empty then $\mathbb{D}$ is called *collegial* and the *Nakamura number* $v(\mathbb{D})$ is defined to be ∞.

(ii) A member M of $\mathbb{D}$ is *minimal decisive* if and only if M belongs to $\mathbb{D}$, but for no member i of M does $M\backslash\{i\}$ belong to $\mathbb{D}$.

(iii) If the collegium $\kappa(\mathbb{D})$ is empty then $\mathbb{D}$ is called *non-collegial* .If $\mathbb{D}'$ is a subfamily of $\mathbb{D}$ consisting of minimal decisive coalitions, with $\kappa(\mathbb{D}') = \subsetneqq$ then call $\mathbb{D}'$ a *Nakumura subfamily* of $\mathbb{D}$.

(iv) Consider the collection of all Nakamura subfamilies of $\mathbb{D}$. Since N is finite these subfamilies can be ranked by their cardinality. Define the *Nakamura number,* $v(\mathbb{D})$, by

$$v(\mathbb{D}) = \min\{|\mathbb{D}'| \colon \mathbb{D}' \subset \mathbb{D} \text{ and } \kappa(\mathbb{D}') = \subsetneqq\}$$

A *minimal non-collegial subfamily* is a Nakamura subfamily, $\mathbb{D}_{\min}$, such that $|\mathbb{D}_{\min}| = v(\mathbb{D})$.

(v) If σ is a BF with $\mathbb{D}_\sigma$ its family of decisive coalitions, then define the *Nakamura number*, $v(\sigma)$, to be $v(\mathbb{D}_\sigma)$, . and say σ is *collegial* or *non-collegial* depending on whether $\mathbb{D}_\sigma$ is collegial or not.

Example 3.2.1. As an example, consider the $q(w)$-rule given by

$$q(w) = [q\colon w_1, w_2, w_3, w_4] = [6\colon 5, 3, 2, 1].$$

We may take $\mathbb{D}_{\min} = \{\{1,4\},\{1,3\},\{2,3,4\}\}$ so $v(\sigma_{q(w)}) = 3$.

For a non-collegial q-rule, σ_q, we can relate $v(\sigma_q)$ to $v(n,q)$.

Lemma 3.2.2.

(i) For any non-collegial voting rule σ, with a society of size n,

$$v(\sigma) \leq n.$$

(ii) For a non-collegial q-rule σ_q,

$$\begin{aligned} v(\sigma_q) &= 2 + v(n,q), \\ \text{so that } v(\sigma_q) &< 2 + q. \end{aligned}$$

(iii) For any *proper* voting rule σ,

$$v(\sigma) \geq 3.$$

(iv) For simple majority rule $\sigma_m, v(\sigma_m) = 3$ except when $(n,q) = (4,3)$ in which case $v(\sigma_3) = 4$.

Proof.

(i) Consider any Nakamura subfamily, $\mathbb{D}'$ of $\mathbb{D}_{\sigma'}$ where $|\mathbb{D}'| = h$ and each coalition M_i in $\mathbb{D}'$ is of size $m_i \leq n-1$. Then $|M_i \cap M_j| \leq n-2$, for any $M_i, M_j \varepsilon \mathbb{D}'$. Clearly $|\kappa(\mathbb{D}')| \leq n-h$. In particular, for the *minimal non-collegial subfamily*, $\mathbb{D}_{\min}$, $h = v(\sigma)$ and $0 = |\kappa(\mathbb{D}_{\min})| \leq n - v(\sigma)$. Thus $v(\sigma) \leq n$.

(ii) For a q-rule, σ_q, let $\mathbb{D}_q$ be its family of decisive coalitions, and let $\mathbb{D}_{\min}$ be a *minimal non-collegial subfamily.* If $M_1, M_2 \varepsilon \mathbb{D}_{\min}$ then $|M_1 \cap M_2| \geq 2q - n$. By induction if $|\mathbb{D}'| = h$ for any $\mathbb{D}' \subset \mathbb{D}_{\min}$ then $|\kappa(\mathbb{D}')| \geq hq - (h-1)n$. Thus $h < \frac{n}{n-q} \Rightarrow |\kappa(\mathbb{D}')| > 0$. Now

$v(n,q) < \frac{q}{n-q}$ so $1 + v(n,q) < \frac{n}{n-q}$. Hence $|\mathbb{D}'| \leq 1 + v(n,q) \Rightarrow |\kappa(\mathbb{D}')| > 0$. Therefore $v(\sigma_q) > 1 + v(n,q)$. On the other hand there exists $\mathbb{D}'$ such that $|\kappa(\mathbb{D}')| = hq - (h-1)n$. Consequently $h \geq \frac{n}{n-q} \Rightarrow |\kappa(\mathbb{D}')| = 0$. But $1 + v(n,q) < \frac{n}{n-q} \leq 2 + v(n,q)$. Thus $h \geq 2 + v(n,q) \Rightarrow |\kappa(\mathbb{D}')| = 0$. Hence $v(\sigma_q) = 2 + v(n,q)$. If $q = n$, then σ_q is collegial. When $q \leq n-1$, clearly

$$v(\sigma_q) = 2 + v(n,q) < 2 + \frac{q}{n-q} \leq 2 + q.$$

Hence $v(\sigma_q) < 2 + q$.

(iii) By definition σ is *proper* when $M_1 \cap M_2 \neq \Phi$ for any $M_1 \cap M_2 \varepsilon \mathbb{D}_\sigma$. Clearly $\kappa(\mathbb{D}') \neq \Phi$ when $|\mathbb{D}'| = 2$ for any $\mathbb{D}' \subset \mathbb{D}_\sigma$ and so $v(\sigma) \geq 3$.

(iv) Majority rule is a q-rule with $q = k+1$ when $n = 2k$ or $n = 2k+1$. In this case $\frac{q}{n-q} = \frac{k+1}{k} = 1 + 1/k$ for n odd or $\frac{k+1}{k-1} = \frac{2}{(k-1)}$ for n even. For n odd $\geq 3, k \geq 2$ and so $v(n,q) = 1$. For n even ≥ 6, $k \geq 3$ and so $1 < \frac{q}{n-q} \leq 2$. Thus $v(n,q) = 1$ and $v(\sigma_m) = 3$., Hence $v(\sigma_m) = 3$ except for the case $(n,q) = (4,3)$. In this case, $k = 2$, so $\frac{q}{n-q} = 3$ and $v(4,3) = 2$ and $v(\sigma_3) = 4$. ∎

Comment 3.2.2 To illustrate the Nakamura number, note that if σ is proper, strong, and has two distinct decisive coalitions then $v(\sigma) = 3$. To see this suppose M_1, M_2 are minimal decisive. Since σ is proper $A = M_1 \cap M_2 \neq \Phi$, must also be losing. But then $N\backslash A\varepsilon\mathbb{D}$ and so the collegium of $\{M, M', N\backslash A\}$ is empty. Thus $v(\sigma) = 3$.

By Lemma 3.2.1, a q-rule maps $A(r)^N \rightarrow A(r)$ iff $r \leq v(n,q) + 1$. By Lemma 3.2.2, this cardinality restriction may be written as $r \leq v(\sigma_q) - 1$. The following Nakamura Theorem gives an extension of the Ferejohn-Grether lemma.

Theorem 3.2.3. (Nakamura, 1978)

Let σ be a voting rule, with Nakamura number $v(\sigma)$. Then $\sigma(p)$ is acyclic for all $p\varepsilon A(r)^N$ iff $r \leq v(\sigma) - 1$.

Before proving this Theorem it is useful to define the following sets.

Definition 3.2.2. Let σ be a BF, with decisive coalitions $\mathbb{D}_\sigma$ and let p be a profile on W.

(i) For a coalition $M \subset N$, define the *Pareto set* for M (at p) to be

$$Pareto(W, M, p) = \{x\varepsilon W : \nexists y\varepsilon W \text{ s.t. } yp_i x \forall i\varepsilon M\}.$$

If $M = N$ then this set is simply called the *Pareto set*.

(ii) The core of $\sigma(p)$ is

$$Core(\sigma, W, N, p) = \{x\varepsilon W : \nexists y\varepsilon W \text{ s.t. } y\sigma(p)x\}.$$

Thus

$$Core(\sigma, W, N, p) \subset \bigcap_{M\varepsilon\mathbb{D}_\sigma} Pareto(W, M, p)$$

with equality if σ is a voting rule.

(iii) An alternative $x\varepsilon W$ belongs to the *cycle set*, $Cycle(\sigma, W, N, p)$, of $\sigma(p)$ in W iff there exists a $\sigma(p)$-cycle

$$x\sigma(p)x_2\sigma(p)\ldots\sigma(p)x_r\sigma(p)x.$$

If there is no fear of ambiguity write $Core(\sigma, p)$ and $Cycle(\sigma, p)$ for the core and cycle set respectively. Note that by Theorem 2.5.2, if $\sigma(p)$ is acyclic on a *finite* alternative set W, so that $Cycle(\sigma, W, N, p)$ is empty, then the $Core(\sigma, W, N, p)$ is non-empty. Of course the choice and cycle sets may both be non-empty. We are now in a position to prove the sufficiency part of Nakamura's Theorem

Lemma 3.2.4. Let σ be a non-collegial voting rule with Nakamura number $v(\sigma)$ on the set W. If $p\varepsilon A(W)^N$ and $Cycle(\sigma, W, N, p) \neq \Phi$ then $|W| \geq v(\sigma)$.

Proof. Since $Cycle(\sigma, W, N, p) \neq \Phi$ there exists a set $Z = \{x_1, \ldots, x_r\} \subset W$ and a $\sigma(p)$-cycle (of length r) on Z:

$$x_1\sigma(p)x_2 \cdots x_r\sigma(p)x_1.$$

Write $x_r \equiv x_0$. For each $j = 1, \ldots, r$, let M_j be the decisive coalition such that $x_{j-1}p_i x_j$ for all $i\varepsilon M_j.Without$ loss of generality we may suppose that all $M_1 \ldots, M_r$ are distinct and minimally decisive and $|W| \geq r$. Let $\mathbb{D}' = \{M_1, \ldots, M_r\}$ and suppose that $\kappa(\mathbb{D}') \neq \Phi$. Then there exists $i\varepsilon\kappa(\mathbb{D}')$ such that

$$x_1 p_i x_2 \cdots x_r p_i x_1.$$

But by assumption, $p_i\varepsilon A(W)$. By contradiction, $\kappa(\mathbb{D}') = \Phi$, and so, by definition of $v(\sigma), |\mathbb{D}'| \geq v(\sigma)$. But then $r \geq v(\sigma)$ and so $|W| \geq v(\sigma)$. ∎

This proves the sufficiency of the cardinality restriction, since if $r \leq v(\sigma) - 1$, then there can be no $\sigma(p)-$cycle for $p \varepsilon A(r)^N$. We now prove necessity, by showing that if $r \geq v(\sigma)$ then there exists a profile $p \varepsilon A(r)^N$ such that $\sigma(p)$ is cyclic.

To prove this we introduce the notion of a *σ-complex* by an example

Example 3.2.2. Consider the voting rule, σ, with six players $\{1, 2, 3, 4, 5, 6\}$ whose minimal decisive coalitions are $\mathbb{D}_{\min} = \{M_1, M_2, M_3, M_4\}$ where $M_1 = \{2, 3, 4\}$, $M_2 = \{1, 3, 4\}$, $M_3 = \{1, 2, 4, 5\}$ $M_4 = \{1, 2, 3, 5\}$. Clearly $v(\sigma) = 4$. We represent σ in the following way. Since $\kappa(\mathbb{D}_{\min} \backslash \{M_j\}) = \{j\}$, for $j = 1, \ldots, 4$, we let $Y = \{y_1, y_2, y_3, y_4\}$ be the set of vertices, and let each y_j represent the players $\{1, ..4\}$. Let Δ be the simplex in $\mathbb{R}^3$ spanned by Y. We define a representation ϕ by $\phi(\{j\}) = y_j$ and $\phi(M_j) = \Delta(Y \backslash \{y_j\})$ for $j = 1, \ldots, 4$.Thus $\phi(M_j)$ is the face opposite y_j. Now player 5 belongs to both M_3 and M_4, but not to M_1 or M_2, and so we place y_5 at the barycenter of the intersection of the faces corresponding to M_3 and M_4. Finally since player 6 belongs to no minimally decisive coalition let $\phi(\{6\}) = \{y_6\}$, an isolated vertex. Thus Δ_σ consists of the four faces of Δ together with $\{y_6\}$. See Figure 1

[Insert Figure 1 here. Caption: The Simplex representing the voting rule]

A representation,ϕ, of σ allows us to construct a profile p, on a set of cardinality $v(\sigma)$ such that $\sigma(p)$ is cyclic. Note that the simplex is situated in dimension $v(\sigma) - 1$. We use this later to construct cycles in dimension $v(\sigma) - 1$.

We now define the notion of a complex.

Definition 3.2.2. A complex Δ.

Let Δ be the abstract simplex in $\mathbb{R}^w$ of dimension $v - 1$, where $v - 1 \leq w$. The simplex Δ may be identified with the convex hull of a set of v district points, or *vertices*, $\{y_1, \ldots, y_v\} = Y$. Opposite the vertex y_j is the *face* $F(j)$ where $F(j)$ is itself a simplex of dimension $(v - 2)$, and may be identified with the convex hull of the $(v - 1)$ vertices $\{y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_v)$. Say Δ is spanned by Y and write $\Delta(Y)$ to denote this. The *edge* of Δ is an intersection of faces. Let $V = \{1, \ldots, v\}$. Then for any subset R of V, define the edge

$$F(R) = \underset{j \varepsilon R}{\cap} F(j).$$

Clearly $F(R)$ is spanned by $\{y_j : j \varepsilon V \backslash R\}$ It is a simplex of dimension $v - 1 - |\mathbb{R}|$ opposite $\{y_j : j \varepsilon R\}$. In particular if $R = 1, \ldots, j - 1, j + 1, \ldots, v\}$ then $F(R) = \{y_j\}$. Finally if $R = V$ then $F(R) = \Phi$. If $\Delta(Y')$ is a simplex spanned by a subset of Y' of Y then the *barycenter* of $\Delta\,(Y')$ is the point

$$\theta(\Delta(Y')) = \frac{1}{|Y'|} \sum_{y_j \varepsilon Y'} y_j.$$

A complex Δ, of dimension $v - 1$, based on the vertices $Y = \{y_1, \ldots, y_n\}$ is a family of simplices $\{\Delta(Y_k) : Y_k \subset Y\}$ where each simplex $\Delta(Y_k)$ in Δ has dimension at most $v - 1$, and the family is closed under intersection, so $\Delta(Y_j) \cap \Delta(Y_k) = \Delta(Y_j \cap Y_k)$.

Given a simplex $\Delta\,(Y)$, where $Y = \{y_1, \ldots, y_v\}$, the natural complex Δ of dimension $(v - 2)$ on $\Delta(Y)$ is the family of faces of $\Delta(Y)$ together with all edges. If $\Delta(Y) \subset \mathbb{R}^w$ for $w \geq v - 1$, then the intersection of all faces of $\Delta(Y)$ will be empty.

Definition 3.2.3.

(i) Let $\mathbb{D}$ be a family of subsets of N, with Nakamura number, v. A *representation* $(\phi, \Delta, \mathbb{D})$ of $\mathbb{D}$ is a complex Δ of dimension $(v - 1)$ in $\mathbb{R}^w$, for $w \geq v - 1$, spanned by $Y = \{y_1, \ldots, y_n\}$ and a bijective correspondence (or *morphism*)

$$\phi\colon\ (\mathbb{D}, \cap) \rightarrow (\Delta, \cap)$$

between the coalitions in $\mathbb{D}_{\min}$ and the faces of Δ, which is natural with respect to intersection. That is to say for any subfamily$\mathbb{D}'$ of $\mathbb{D}$,

$$\phi\colon\ (\kappa\,(\mathbb{D}')) = \cap_{M\varepsilon\mathbb{D}'}\phi\,(M)$$

Moreover, ϕ can be extended over N :if for some $i\varepsilon N$, $\mathbb{D}_i = \{M\varepsilon\mathbb{D}\colon\ i\varepsilon M\} \neq \Phi$ then $\phi(\{i\}) = \theta\,(\phi\,(\kappa\,(\mathbb{D}_i)))$, whereas if $\mathbb{D}_i = \Phi$ then $\phi(\{i\})$ is an isolated vertex in Δ.

(ii) If there exists a representation $(\phi, \Delta, \mathbb{D})$ of $\mathbb{D}$ then denote Δ by $\Delta\,(\mathbb{D})$.

(iii) If σ is a BF with decisive coalitions $\mathbb{D}_\sigma$ and $(\phi, \Delta, \mathbb{D}_\sigma)$ is a representation of $\mathbb{D}_\sigma$ then write Δ as Δ_σ and say Δ_σ is the the σ-*complex* which represents σ.

Schofield (1984b) has shown the folowing.

Theorem 3.2.4. Let $\mathbb{D}$ be a family of subsets of N, with Nakamura number $v < \infty$. Let $\mathbb{D}_{\min}$ be a minimal non-collegial subfamily of $\mathbb{D}$. Then there exists a simplex $\Delta(Y)$, in $\mathbb{R}^{v-1}$, spanned by $Y = \{y_1, \ldots, y_v\}$ and a representation $\phi\colon\ (\mathbb{D}_{\min}, \cap) \to (\Delta, \cap)$ where Δ is the natural complex based on the faces of $\Delta(Y)$. Furthermore:

(i) There exists a subset $V = \{1, \ldots, v\}$ of N such that, for each $j\varepsilon V, \phi(\{j\}) = y_j$, a vertex of $\Delta(Y)$.

(ii) After relabelling, for each $M_j\varepsilon\mathbb{D}_{\min}$,

$$\phi(M_j) = F(j)$$

the face of $\Delta(Y)$ opposite y_j.

Proof. Each proper subfamily $\mathbb{D}_t = \{.., M_{t-1}, M_{t+1}, ..\ :\ t = 1, ...v\}$ of $\mathbb{D}_{\min}$ has a non-empty collegium, $\kappa(\mathbb{D}_t)$, and each of these can be identified with a vertex,y_t of Δ.If $j\varepsilon.\kappa(\mathbb{D}_t)$,then j is assigned the vertex y_t. Continue by induction: if $j\varepsilon\kappa(\mathbb{D}_t \cap \mathbb{D}_s) - \kappa(\mathbb{D}_t) - \kappa(\mathbb{D}_s)$ then j is assigned the barycenter of $[y_t, y_s]$ .By this method we assign a vertex to each member of the set $N(\mathbb{D}_{\min})$ consisting of those individuals who belong to at least one coalition in $\mathbb{D}_{\min}$. This assignment gives a representation $(\phi, \Delta, \mathbb{D}_{\min})$.

Corollary 3.2.5. Let σ be a voting rule with Nakamura number $v(\sigma)$. Then there exists a σ-complex $\Delta_{\sigma'}$, of dimension $v(\sigma) - 2$ in $\mathbb{R}^w$, for $w \geq v(\sigma) - 1$, which represents σ.

Proof. Let $v(\sigma) = v$. Let $\mathbb{D}_{\min}$ be the minimal non-collegial subfamily of $\mathbb{D}_\sigma$ and let $\phi\colon\ (\mathbb{D}_{\min}, \cap) \to (\Delta, \cap)$ be the representation constructed in Theorem 3.2.4. Extend ϕ to a representation $\phi\colon\ \mathbb{D}_\sigma \to \Delta\,(\mathbb{D}_\sigma)$ by adding new faces and vertices as required. Finally, the complex Δ_σ can be constructed so that, for any $\mathbb{D}' \subset \mathbb{D}_{'}$ then $\kappa\,(\mathbb{D}') = \Phi$ if and only if $\cap\phi\,(M_j) = \Phi$, where the intersection is taken over all $M_j\varepsilon\mathbb{D}'$. ∎

We are now in a position to prove the necessity part of Nakamura's Theorem.

Corollary 3.2.6. Let σ be a voting rule, with Nakamura number $v(\sigma) = v$, on a finite alternative set W. If $|W| \geq v(\sigma)$ then there exists an acyclic profile p on W such that $Cycle(\sigma, W, N, p) \neq \Phi$ and $Core(\sigma, W, N, p) = \Phi$.

Proof. Construct a profile $p\varepsilon A(W)^N$ and a $\sigma(p)$ cycle on W. as follows. Each proper subfamily $\mathbb{D}_t = \{.., M_{t-1}, M_{t+1}, ..\ :\ t = 1, ...v\}$ of $\mathbb{D}_{\min}$ has a non-empty collegium, $\kappa(\mathbb{D}_t)$. By Theorem 3.2.5, each of these can be identified with a vertex,y_t of Δ_σ. Without loss of generality we relabel so that $Y = \{y_1, ...y_t\ ..y_v\} \subset W$. Let $V = \{1, ...v\}$.. We assign preferences to the members of these collegia on the set Y as follows.

$$\begin{array}{cccc} p_1 : \kappa(\mathbb{D}_1) & p_2 : \kappa(\mathbb{D}_2) & \ldots & p_v : \kappa(\mathbb{D}_v) \\ y_1 & y_2 & & y_v \\ y_2 & y_3 & & y_1 \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ y_v & y_1 & \ldots & y_{v-1} \end{array} .$$

To any individual $j\varepsilon\ N(\mathbb{D}_{\min})$ who is assigned a position at the barcenter, $\theta(\Delta(Y'))$, for a subset $Y' = \{y_r; r\varepsilon R \subset V\}$, we let $p_j = \underset{j\varepsilon R}{\cap} p_r$.

It follows from the construction that every member j of coalition M_t has a preference satisfying

$$\underset{r \neq t}{\cap} p_r. \subseteq p_j.$$

It then follows that each $j\varepsilon M_t$ has the preference $y_{t+1}p_jy_t$, where we adopt the notational convention that $y_{v+1} = y_1$.

We thus obtain the cycle

$$y_1\sigma(p)y_v \cdots \sigma(p)y_2\sigma(p)y_1.$$

This profile can be extended over Y by assigning to an individual j not in $N(\mathbb{D}_{\min})$ the preference of complete indifference. Obviously $Cycle(\sigma, W, N, p) \neq \Phi$ and $Core(\sigma, W, N, p) = \Phi$.The argument obviously holds whenever $|W| > v(\sigma)$,again by assigning indifference to alternatives outside Y. ▮

Example 3.2.3.

To illustrate the construction, consider Example 3.2.2 The profile constucted according to the Corollary is:

1	2	3	4
y_1	y_2	y_3	y_4
y_2	y_3	y_4	y_1
y_3	y_4	y_1	y_2
y_4	y_1	y_2	y_4

Because y_5 lies on the arc $[y_{1,}y_2]$ in the Figure, we define $p_5 = p_1 \cap p_2$, so

$$y_2p_5y_3p_5y_4I_5y_1.$$

Individual 6 is assigned indifference.

$$y_1I_6y_2I_6y_3I_6y_4.$$

For this "permutation profile" we observe the following:

(i) $M_1 = \{2, 3, 4\}$ and$y_4p_iy_1$ for $i\varepsilon M_1$.

(ii) $M_2 = \{1, 3, 4\}$ and $y_1p_iy_2$ for $i\varepsilon M_2$.

(iii) $M_3 = \{1, 2, 4, 5\}$ and $y_2 p_i y_3$ for $i \varepsilon M_3$.

(iv) $M_4 = \{1, 2, 3, 5\}$ and $y_3 p_i y_4$ for $i \varepsilon M_4$.

Thus we obtain the cycle

$$y_1 \sigma(p) y_2 \sigma(p) y_3 \sigma(p) y_4 \sigma(p) y_1.$$

Clearly $Cycle(\sigma, p) = \{y_1, y_2, y_3, y_4\} = GO(N, p)$ and $Core(\sigma, p) = \Phi$.
Lemma 3.2.4 and Corollary 3.2.6 together prove Nakamura's Theorem.

A natural preference to use is *Euclidean preference* defined by xp_iy if and only if $||x - x_i|| < ||y - x_i||$, for some bliss point, x_i, in W, and norm $|| - ||$ on W. Clearly Euclidean preference is convex. $Pareto(W, M, p) = Con\{x_i : i \varepsilon M\}$.Moreover, for any voting rule, σ, we know tht

$$Core(\sigma, W, N, p) = \bigcap_{M \varepsilon \mathbb{D}_\sigma} Pareto(W, M, p)$$

It follows that if we regard the the simplex constructed in the proof of Corollary 3.2.6 as a Euclidean profile, then for each $M_t \; \varepsilon \mathbb{D}_{\min}$, $Pareto(W, M_t, p) = Con\{y_i : i \varepsilon M_t\}$. But these convex sets correspond to the faces of Δ_σ.Since the faces do not intersect, we see that

$$Core(\sigma, W, N, p) = \Phi.$$

We have therefore proved the following:
Corollary 3.2.7. Let σ be a voting rule, with Nakamura number $v(\sigma)$, on a compact convex subset,W, of $\mathbb{R}^w$. If $w \geq v(\sigma) - 1$, then there exists an Euclidean profile p on W such that $Cycle(\sigma, W, N, p) \neq \Phi$ and $Core(\sigma, W, N, p) = \Phi$.

Schofield (1984b) and Strnad (1985) also proved the following.
Theorem 3.2.7. Let σ be a voting rule, with Nakamura number $v(\sigma) = v$, on a compact convex subset of $\mathbb{R}^w$. If $w < v(\sigma) - 1$, then for any Euclidean profile p on W $Core(\sigma, W, N, p) \neq \Phi$ and $Cycle(\sigma, W, N, p) = \Phi$

Thus the Nakamura number of a voting rule classifies the behavior of the rule.

4.Conclusion

Although these social choice results demonstrate that majority rule can be unstable, this only occurs of the number of alternatives, or the dimension of the space is sufficiently large. Empirical models of elections (Schofield and Sened, 2006) suggest that the underlying policy space is nearly always of two dimensions. Thus in polities based on methods of proportional representation, there need not be a poicy core point. Instead outcomes can be expected to lie within a central domain of policy, bounded by the locations of some of the parties.

Acknowledgement. This article is based on research supported by NSF Grant SES 024173

Glossary

Binary social preference function : a social choice function based on binary comprison.
Choice : a method of choosing socially preferred elements.
Condorcet Cycle : a voting cycle on three possibilities.
Core :set of chosen alternatives.
Decisive Coalition :a winningcoalition under some rule.
Dictator : an individual able to choose the social outcome.
Impossibility Theorem :the result that a "rational rule" must have a dictator.
Plurality Rule : vote method where the largest colition wins.
Simple Voting Rule : social choice rule using a specified set of winning coalitions.

Bibliography

Armstrong T.E. (1980) Arrow's theorem with restricted coalition algebras. *Journal of Mathematical Economics* **7**, 55–75.[Arrow's theorem with a continuum of voters].

Arrow K.J. (1951) *Social Choice and Individual Values*. New York: Wiley. [First major result in social choice]

Arrow K.J. (1959) Rational choice functions and orderings. *Economica* **26**, 121–127. [The theorem applied to choice functions]

Black D. (1958) The theory of committees and elections. Cambridge: Cambridge University Press. [One dimensional core]

Brown D.J. (1973) Acyclic choice. Mimeo, Cowles Foundation, Yale University. [Acyclic choice and the collegium]

Brown D.J. (1975 Aggregation of preferences. *Quarterly Journal of Economics* **8**, 471–452. [Acyclic choice and the collegium]

Cox G. (1984) Non-collegial simple games and the nowhere denseness of the set of preference profiles having a core. *Social Choice and Welfare* **1**, 159–164. [Perturbation of the core]

Fan K. (1961) A generalization of Tychonoff's fixed point theorem. *Math Annalen* **42**, 305-310. [The extension of Tychonoff's Theorem].

Ferejohn J.A (1977) Decisive coalitions in the theory of social choice. *Journal of Economic Theory* **12**, 301-307. [Choice funtions].

Ferejohn J.A, Fishburn P.C. (1979) Representations of binary decision rules by generalized decisiveness structures. *Journal of Economic Theory* **21**, 28–45. [Rationality properties of choice].

Ferejohn J.A, Grether D.M. (1974) On a class of rational decision procedures. Journal of Economic Theory **8**, 471–482. [Choice with resticted set of alternatives].

Ferejohn JA, Grether DM (1977) Weak path independence. Journal of Economic Theory **14**,19–31.(Rationality properties of choice].

Ferejohn J.A., Grether D.M., McKelvey R.D. (1982) Implementation of democratic social choice functions. *Review of Economic Studies* **49**,439–446.[Rationality properties of choice].

Fishburn P.C. (1970) Arrow's impossibility theorem: Concise proof and infinite voters. *Journal of Economic Theory* **2**, 103–106.(Arrow's theorem with a continuum of voters)

Fishburn P.C. (1973) *The Theory of Social Choice*. Princeton: Princeton University Press. (Rationality properties of choice

Gibbard A. (1969) Intransitive social indifference and the Arrow dilemma. Mimeo, Department of Philosophy, University of Michigan.[Arrow's theorem with an oligarchy].

Golubitky M. Guillemin V. (1973) *Stable Mappings and their singularities* New York: Springer Verlag. [Singularity theory].

Greenberg J. (1979) Consistent majority rules over compact sets of alternatives. *Econometrica* **47**, 627–636. [A core in restricted dimension]..

Hanssen B. (1976) The existence of group preference functions. *Public Choice* 28, 89–98.[Arrow's theorem with an oligarchy.

Hirsch M. W. (1976) *Differential Topology*. New York: Springer Verlag. [Singularity theory]

Inada K. (1969) On the simple majority decision rule. *Econometrica* **37**, 490-506. [Majority rule].

Kirman A.P., Sondermann D. (1972) Arrow's impossibility theorem, many agents and invisible dictators. *Journal of Economic Theory* **5**, 267–278. [Arrow's theorem with an invisible dictator].

Kramer G. H. (1973) On a class of equilibrium conditions for majority rule. *Econometrica* **41**, 285-297. [The existence of a Condorcet Cycle]

Kramer G. H., Klevorick A. K. (1974) Existence of a ‘local’ cooperative equilibrium in a class of voting games. *Review of Economic Studies* **41**, 539–548. [The existence of a local equilibrium].

Maskin E. (1977) Nash equilibrium and welfare optimality. Mimeo, Department of Economics, MIT.(Nash equilibrium in voting)

Maskin E (1979) Implementation and strong Nash equilibrium. *Aggregation and Revelation of Preferences*(ed. J.J.Laffont . New York: North Holland, [Nash equilibrium in voting].

McKelvey R. D. (1976) Intrasitivities in multidimensional voting models and some implications for agenda control. *Journal of Economic Theory* **12**, 472–482. [Existence of voting cycles].

McKelvey R. D. (1979) General conditions for global intransitivities in formal voting models. *Econometrica*. *47*,1085–1111. [Existence of voting cycles]

McKelvey R.D., Wendell R.E. (1976) Voting equilibria in multidimensional choice spaces. *Mathematics of Operations Research* **1**,144-158. [Existence of equilibrium]

McKelvey R. D., Schofield N. (1986) Structural instability of the core. *Journal of Mathematical Economics*. **15**, 179–198.[Structural instability of the core]

McKelvey R. D., Schofield N. (1987) Generalized symmetry conditions at a core point. *Econometrica.* 55, 923–933.[Generalized symmetry conditions at a core point.].

Nakamura K. (1979) The vetoers in a simple game with ordinal preference. *International Journal of Game Theory* **8**, 55–61.[The Nakamura number and existence of a core].

Plott C. R. (1967) A notion of equilibrium and its possibility under majority rule. *American Economic Review* **57**, 787–806.[Symmetry conditions at the core].

Plott C.R. (1970) Recent results in the theory of voting. *Frontiers of quantitative economics* (ed. M.D. Intrilligator MD). Amsterdam.: North Holland. .[Choice function properties].

Plott C.R. (1973) Path independence, rationality and social choice. *Econometrica* **41**, 1075–1091.[Choice function properties].

Plott C. R. (1976) Axiomatic social choice theory: An overview and interpretation. *American Journal of Political Science* 20, 511–594.[Choice function properties].

Rubinstein A. (1979) A note about the nowhere denseness of societies having an equilibrium under majority rule. *Econometrica* **47**, 511–514. [Instability of a core under non-convex perturbation].

Saari D. (1997) The Generic Existence of a Core for q rules, *Economic Theory* **9**, 219–260. [Instability of the core in high dimension].

Schmitz N. (1977) A further note on Arrow’s impossibility theorem. *Journal of Mathematical Economics* **4**, 189–196.[Arrow’s theorem with a continuum of voters].

Schofield N. (1983) Generic instability of majority rule. *Review of Economic Studies* **50**, 695–705.[Generic instability of majority rule].

Schofield N. (1984a) Social equilibrium and cycles on compact sets. *Journal of Economic Theory* **33**, 59–71.[Existence of core in low dimension].

Schofield N. (1984b) Classification theorem for smooth social choice on manifold. Social Choice and Welfare.**1,** 187-210.[Classification theorem].

Schofield N. (1985) Social Choice and Democracy. New York: Springer Verlag. [Structural stability of core].

Schofield N. (2003). *Mathematical Methods in Economics and Social Choice.* New York: Springer Verlag. .[Proof of the Fan theorem].

Schofield N. , Sened.I. (2006) *Multiparty Democracy: Elections and Legislative Politics.* New York: Cambridge University Press. [Applications of social choice theory to democratic elections].

Schwartz T. (1976) Choice functions, rationality conditions and variations on the weak axiom of revealed preference. *Journal of Economic Theory* **13**, 414–427.[Choice functions].

Schwartz T. (1986) *The Logic of Collective Action.* New York: Columbia University Press. [Choice functions].

Sen A. K. (1966) A possibility theorem on majority decisions. *Econometrica* **34**, 491–499. [Restrictions on preference].

Sen A. K. (1970) Collective choice and social welfare. Amsterdam: North Holland. [The pareto rule].

Sen A. K. , Pattanaik P. K. (1969) Necessary and sufficient conditions for rational choice under majority decisions. Journal of Economic Theory **1**, 178–202. [Restrictions on preference].

Strnad J. (1981 The structure of continuous neutral monotonic social functions. Social Choice and Welfare **2**, 181-195.[Existence of core in low dimension].

Tullock G. (1967) The general irrelevance of the general impossibility theorem. *Quarterly Journal of Economics* **81**, 256-270.[Argument that cycles will be Paretian].

Walker M. (1977) On the existence of maximal elements *Journal of Economic Theory* **16**, 470-474.[A choice with acyclic preference].

Biographical Sketch

Norman Schofield is Director of the Center in Political Economy and William R. Taussig Professor of Political Economy at Washington University in Saint Louis.. He is also Professor of Political Science. He was born in Rothesay, Scotland in 1944. After taking B.Sc degrees in physics and mathematics, in 1965 and 1966, from Liverpool University, he obtained his PhDs in Government and then Economics from Essex University in 1976 and 1985, and later a PhD and a LittD from Liverpool University in 1986. In 1991 he was awarded the Doctorat d'Etat en Science Economique from the University of Caen.

He is currently working on topics in the theory of social choice, political economy, and democracy. His recent publications in democratic theory include articles in the Journal of Theoretical Politics (2005), the European Journal of Political Research (2005), British Journal of Political Science (2005), Political Studies(2003), Politics and Society (2002) and the American Political Science Review (2003). He has articles forthcoming in the Review of Economic Studies, Mathematical Social Science and the American Journal of Political Science. He is a member of the editorial board of Social Choice and Welfare, and has published a number of articles in this journal. He has written several books, including Multiparty Government (with Michael Laver, in 1990), Social Choice and Democracy (1985), and co-edited three volumes: Political Economy: Institutions, Information, and Representation in 1993 , Social Choice, Welfare and Ethics, in 1995 (both with Cambridge University Press), and Collective Decision Making (Kluwer, 1996). His book, Mathematical Methods in Economics and Social Choice, was published by Springer in 2003. In 2006 he published Architects of Political Change as well as Multiparty Democracy (with Itai Sened) both with Cambridge University Press.

Dr. Schofield was elected to the American Academy of Arts and Sciences in April 2005.