Annals of Operations Research 109, 193–211, 2002

Local Nash Equilibrium in Multiparty Politics

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Abstract. We present a model of multi-party, "spatial" competition under proportional rule with both electoral and coalitional risk. Each party consists of a set of delegates with heterogeneous policy preferences. These delegates choose one delegate as leader or agent. This agent announces the policy declaration (or manifesto) to the electorate prior to the election. The choice of the agent by each party elite is assumed to be a local Nash equilibrium to a game form $\tilde{g}$. This game form encapsulates beliefs of the party elite about the nature of both *electoral risk* and the post-election *coalition bargaining game*. It is demonstrated, under the assumption that $\tilde{g}$ is smooth, that, for almost all parameter values, a locally isolated, local Nash equilibrium exists.

In the final section of the paper some empirical work is reviewed in order to obtain some insights into why parties do not simply converge to an electoral center in order to maximize expected vote shares.

1. Introduction

Almost all models of the game of political competition make particular assumptions about the objective functions of the candidates, or parties. The conclusion of these models typically is that these political agents tend to cluster, in equilibrium, near the electoral center. In the simplest "Euclidean" model the strategy space, or policy space, Z is a subset of Euclidean space, $\Re^w$. If there are p different parties competing, and their policy positions are given by the p-vector $(z_1, \dots, z_p) \in Z^p$, then the choice of voter i is usually assumed to be generated by the p-vector

$$\bar{u}_i(z) = \lambda + \varepsilon - A(x_i, z) + B\gamma_i = \big(\bar{u}_{i1}(z), \dots, \bar{u}_{ip}(z)\big). \tag{1}$$

Here $x_i \in Z$ is voter i's "bliss" point, $A(x_i, z)$ is a p-dimensional quadratic form, whose jth entry, $A_j(x_i, z) \equiv A_j(x_i, z_j)$, is assumed to be of the form $\beta_j \|x_i - z_j\|^2$. The vector γ_i is an individual specific d-vector, while B is a $p \times d$ matrix of coefficients, describing the relationship between individual characteristics and voting propensity. The p-vector, ε, is a stochastic random variable, each entry ε_j being drawn from a smooth distribution, generally Gaussian. The behavioral choice of voter i is described by a p-vector, c_i. It is usually assumed that the entry $c_{ij} = 1$ if and only if $\bar{u}_{ij}(z) > \bar{u}_{ik}(z)$ for all $k \neq j$ and $c_{ij} = 0$ if $\bar{u}_{ij}(z) < \bar{u}_{ik}(z)$ for some $k \neq j$. The realized vote share, $v_j(z)$, for party j is then $\frac{1}{n}\sum_i c_{ij}$, where n is the size of the finite voting population, denoted N. Since ε is a random vector, so is $\bar{u}_i(z)$, and thus $v_j(z)$. Using $\rho_{ij}(z)$ to denote the probability that voter i chooses j $(c_{ij} = 1)$ then the expectation $\mathcal{E}(v_j(z))$ will be $\frac{1}{n}\sum_i \rho_{ij}(z)$.

The standard assumption of this stochastic vote model is that each party, j, chooses z_j so as to maximize $\mathcal{E}(v_j(z))$, given the positions of the other parties. A pure strategy Nash equilibrium (PSNE) is a vector $z^* \in Z^p$ such that for each z_j^* it is the case that there exists no other z_j ($\neq z_j^*$) in Z with the property that

$$\mathcal{E}\big(v_j(z_1^*, \ldots, z_j, \ldots, z_p^*)\big) > \mathcal{E}\big(v_j(z_i^*, \ldots, z_j^*, \ldots, z_p^*)\big).$$

Typical assumptions, on the quasi-concavity (in z_j) and joint continuity (in z) of this objective function, lead to an existence proof of PSNE. Moreover, additional assumptions of concavity (with respect to z_j) lead to the conclusion that the PSNE is unique [5]. It is usual in applications of this model to assume symmetry in the voter utility functions ($A_j = A_k$ for $j \neq k$ and $B \equiv 0$). Under certain assumptions on the variance terms of the disturbance ε it can be shown [21] that not only are all pure strategy equilibrium positions identical, but they are at the mean of the voter distribution (at the point, $\bar{x} = \frac{1}{n} \sum x_i$).

Deterministic models of voting ignore the stochastic terms, and set $\varepsilon \equiv 0$. In the earlier two party spatial models it was assumed that each party attempted to gain more votes than the other. For this two party case, define $c_{ij} = 1$ and $c_{ik} = 0$ when $\bar{u}_{ij}(z) > \bar{u}_{ik}(z)$, and $c_{ij} = c_{ik} = \frac{1}{2}$ if $\bar{u}_{ij}(z) = \bar{u}_{ik}(z)$.

Again, the vote share $v_j(z)$ of party j is $\frac{1}{n} \sum_i c_{ij}$. The utility functions of the two parties in this deterministic model are then defined to be $U_j(z) = 1$ and $U_k(z) = -1$ if $v_j(z) > v_k(z)$, while $U_j(z) = U_k(z) = 0$ if $v_j(z) = v_k(z)$. However, continuity of the vote share functions clearly fails, and PSNE generally do not exist [9]. In the Euclidean case, since voter utilities are smooth (in z), certain conditions on the gradients of voter utilities at a point, x, are necessary for the point to be an equilibrium for both parties [24]. However, with certain measurability conditions on the party utility functions (U_j, U_k), defined with respect to the Borel σ-algebra over Z^2, it can be shown that mixed strategy Nash equilibria (MSNE) do exist [6]. The concept of MSNE assumes that each party randomises across some domain in Z. Indeed it can be shown that the support, in Z, of the MSNE is a subset of the so-called "uncovered set" induced by the underlying social preference correspondence, Q, say, on Z [6,11,23].

In the two party case, we may say that a party position, z_j, is socially preferred to a party position z_k (written $z_j \in Q(z_k)$) iff $v_j(z_j, z_k) > v_k(z_j, z_k)$. It can readily be shown that the uncovered set is typically a very small domain in the center of the electoral distribution of voter bliss points. (The definition of the uncovered set in Z, induced by a social preference correspondence, Q, is given in section 3.)

For deterministic party competition with many parties ($p \geqslant 3$), it was unclear how precisely to develop the model. Eaton and Lipsey [16] simply assumed that party utility functions were identified with vote shares, and showed that, as long as the space Z was two-dimensional, then there could be no PSNE. However, if MSNE do exist in multiparty models (under the vote share maximizing assumption), then they will tend to be clustered near the center of the electoral distribution.

In contrast, empirical political scientists have long argued that the "center is empty in politics..." [13,15]. There seems, therefore, to be a contradiction between the theoretical results, and political observation.

One model that has been offered that does not lead to party "convergence" is due to Cox [10]. Cox considered two-party competition. Now, let $z = (z_1, z_2)$ represent the two party positions declared (by manifestos) to the electorate. In this situation there are only three electoral possibilities, which we shall denote by $\mathcal{D}_0$, $\mathcal{D}_1$, and $\mathcal{D}_2$.

Under $\mathcal{D}_0$, the vote shares $v_1(z_1, z_2)$ and $v_2(z_1, z_2)$ are identical. With this election result, the governmental outcome is a lottery denoted $\tilde{g}_0(z)$. This policy outcome is a randomization between z_1 and z_2 (where parties compromise over policy) and both parties share a government perquisite, receiving δ_1, δ_2 respectively. The electoral result $\mathcal{D}_1$ denotes that party 1 wins $(v_1(z_1, z_2) > v_2(z_1, z_2))$, implements policy z_1 and receives a perquisite $\delta_1 + \delta_2$. Similarly $\mathcal{D}_2$ denotes a win by party 2, which implements z_2 and also receives $\delta_1 + \delta_2$.

Cox assumed the electoral outcome was stochastic. The probabilities of the electoral outcomes $\mathcal{D}_0$, $\mathcal{D}_1$, and $\mathcal{D}_2$ can be denoted by $\pi_0(z)$, $\pi_1(z)$ and $\pi_2(z)$. Then the utility function of party 1, say, can be written

$$U_1(z) = \pi_0(z)\widetilde{V}_1\big(\tilde{g}_0(z)\big) + \pi_1(z)V_1(z_1, \delta_1 + \delta_2) + \pi_2(z)V_1(z_2, 0).$$

Cox also assumed that $V_1(z_1, \delta_1 + \delta_2) = \delta_1 + \delta_2 - \|z_1 - y_1\|^2$, etc.

The points $y_1, y_2 \in Z$ were assumed to be the bliss points of the two party leaders. Here $\widetilde{V}_1$ denotes the extension of V_1, whose domain is the set of lotteries, such as $\tilde{g}_0(z)$, endowed with an appropriate topology. Cox argued that natural assumptions, such as concavity and continuity, would generate existence of PSNE in such a two-party game. Moreover, he gave illustrations of the formal model to suggest that when PSNE did exist, the parties need not adopt identical, or even centrist, positions. Cox's later work considered the various characteristics of an electoral system leading to divergence or convergence [12].

2. Modelling the game form

In this paper we propose a generalization of Cox's model to the case $p \geqslant 3$. To develop the model we make a number of assumptions which are not usual in the literature.

(1) First of all, we assume, with Cox, that parties are concerned about policy. This generally implies that quasi-concavity of the party utility functions will fail. Consequently PSNE will usually not exist. However, we reject the standard focus on MSNE, since it is deemed implausible that parties can randomize over pre-election policy promises.

(2) We base the voter calculus on equation (1). Section 4 below will examine in more detail the relationship between the voter model and the probability function π. In the voter model, we assume ε is drawn from a smooth Gaussian distribution, which is smooth. Since we also assume that the quadratic form A_j is differentiable in z_j, we proceed to argue that the party utility functions $\{U_j\}$ are also differentiable in z_j. This

allows us to assert that there will exist a "critical point" $z = (z_1, \ldots, z_p) \in Z^p$ of $U = (U_1, \ldots, U_p)$. Moreover, transversality arguments allow us to infer that there will exist, almost always, a critical point, z, which is also a local maximum, for each U_j. Such a point we term a Local Nash Equilibrium (LNE).

(3) In models, such as the one presented by Cox, where parties have policy preferences, it is difficult to sustain the argument that a winning party will, after the election, implement its declared position, z_j, rather than its preferred position, y_j. We deal with this difficulty by logically separating the choice of the policy position, z_j, to declare to the electorate, from the preferred policy position of the party, y_j.

We assume that the jth party consists of a set of elite party members, possibly with heterogeneous preferences. These elite party members select, by whatever internal decision making process is deemed appropriate, one of their own to be the party *principal*. If a policy outcome χ is adopted, and party j receives a perquisite, δ_j, then the utility payoff to this principal is given by the function V_j, where

$$V_j(\chi, \delta_j) = -A_j(y_j, \chi) + \alpha_j \delta_j. \tag{2}$$

That is, y_j, is the preferred policy position of the principal. More generally, let $W = Z \times \Delta_p$, where Δ_p is the $(p-1)$-dimensional simplex, denoting the space of all shares of perquisites to the p different parties. Then the utility function of the pth principal can be regarded as a function $V_j : W \to \Re$. Moreover, since A_j is a smooth quadratic form, so is V_j. We extend V_j to $\widetilde{W}$, the space of all probability measures on W. That is to say, $\widetilde{W}$ is endowed with its Borel σ-algebra and with the weak topology [28]. This extension we denote by $\widetilde{V}_j : \widetilde{W} \to \Re$, where, by implication, $\widetilde{V}_j$ is measurable with respect to the Borel σ-algebra on W.

(4) The principal of each party, j, chooses a *leader* for the party. The leader of party j has a preferred policy position, z_j, and utility function of the form

$$V'_j(\chi, \delta_j) = -A_j(z_j, \chi) + \alpha_j \delta_j. \tag{3}$$

Without loss of generality we can assume for the moment that the p-vector $(\alpha_1, \ldots, \alpha_p) = \alpha \in \Re^p$ is known and fixed. Thus the choice of the list of party leaders can be expressed as a smooth "leadership" function

$$q^\alpha : Z^p \to Q^*(W)^P,$$

where $Q^*(W)^P$ is the space of "smooth" convex preference profiles, on W, endowed with the C^1-topology. (This is defined more fully in the technical section in section 3.) Thus $q^\alpha(z)$ is the preference profile of the party leaders. It is assumed, after the principal of party j has selected the party leader (and thus z_j), that the party leader attempts to implement this policy position. Because the leader is chosen in this way, the electorate believes that party j is committed to policy z_j.

(5) Since there may be many parties, and no one party may win the election, the post election states are more complicated than in the two party case of Cox. We assume that the only relevant aspect of the election is the nature of the decisive coalition structure holding after the election. Thus, let $\mathcal{D}_t$ be a possible family of "decisive" coalitions made

up of various parties. Each decisive coalition in $\mathcal{D}_t$ controls a majority of the legislature. We let $\{\mathcal{D}_1, \ldots, \mathcal{D}_t, \ldots, \mathcal{D}_T\}$ be the collection of all such families. Note that each election outcome defines a single family, $\mathcal{D}_t$, say. We assume that pre-election beliefs of all party principals about the election are characterized by a function $\pi : Z^p \to \Delta_T$.

Here $\pi(z) = (\pi_1(z), \ldots, \pi_t(z), \ldots, \pi_T(z))$ where $\pi_t(z)$ denotes the common belief (the probability) that the decisive structure, $\mathcal{D}_t$, will result from the election, when the vector of party leaders' positions is given by $z \in Z^p$. (We use Δ_T to denote the $(T-1)$-dimensional simplex.)

We now begin to list our assumptions:

Assumption 1. The strategy space Z is a compact convex smooth subset of $\Re^w$.

Assumption 2. The electoral map $\pi : Z^p \to \Delta_T$ is C^1-differentiable with respect to the Euclidean topologies on Z^p and Δ_T.

The party principal will be motivated to choose a leader who can both appeal to the electorate and bargain effectively with other leaders over policy. To do this the principal must form a belief about the nature of the bargaining game in each of the various electoral outcomes. We follow Banks and Duggan [4] in assuming that, given the leadership profile $q^\alpha(z)$ and the post election decisive structure $\mathcal{D}_t$, then the outcome is a lottery $\tilde{g}_t^\alpha(z) \in \widetilde{W}$. Such a lottery is a list of coalitions, specifying to each coalition a probability of occurrence, a policy outcome (or set of outcomes) and a list of perquisites, one for each coalition member. As a function, we write $\tilde{g}_t^\alpha : Z^p \to \widetilde{W}$. We shall assume that $\tilde{g}_t^\alpha$ is smooth. To specify what this means, consider any measurable utility $\widetilde{V} : \widetilde{W} \to \Re$. There is a restriction operation, $\widetilde{\ } : \widetilde{V} \to \widetilde{\widetilde{V}}$ where $\widetilde{\widetilde{V}} : W \to \Re$, is obtained by restricting $\widetilde{V}$ to $W \subset \widetilde{W}$.

Say $\widetilde{V} : \widetilde{W} \to \Re$ is *smooth* if it is both measurable and continuous with respect to the weak topology on $\widetilde{W}$, and moreover $\widetilde{\widetilde{V}} : W \to \Re$ is C^1-differentiable on its domain. Let $\widetilde{\mathcal{U}}$ denote the set of such smooth functions on $\widetilde{W}$. If $g \in \widetilde{W}$ is a measure on the Borel σ-algebra of W, then $\widetilde{V}(g) = \int \widetilde{V}\, \mathrm{d}g$ is well defined. We say a function $g : Z^p \to \widetilde{W}$ is *smooth* iff the composite function

$$U^g : Z^p \to \Re$$

given by $U^g(z) = \widetilde{V}(g(z))$ is C^1-differentiable in z for all $\widetilde{V} \in \widetilde{\mathcal{U}}$.

Assumption 3. For fixed α, and under the belief that the election outcome will be $\mathcal{D}_t$, and given the vector z of leaders' positions, then the party principals believe that the outcome will be a lottery $\tilde{g}_t^\alpha(z)$ (a measure in the Borel σ-algebra on W). Moreover, $\tilde{g}_t^\alpha : Z^p \to \widetilde{W}$ is smooth.

We can now define the game form $\tilde{g}$ for party competition.

Assumption 4. The game form $\tilde{g}$, at the parameter α, is represented by $\tilde{g}(\alpha) = \{\tilde{g}_t^\alpha, \pi_t\}_{t=1,\dots,T}$, where each $\tilde{g}_t^\alpha$ is smooth and

$$\pi = (\pi_1, \dots, \pi_p) : Z^p \to \Delta_T$$

is a C^1-differentiable electoral map. More briefly we say $\tilde{g}$ is smooth.

This generates a game $(\tilde{g}, U^\alpha)$, where $U^\alpha : Z^p \to \Re^p$ is the C^1-differentiable profile map. The jth component of U^α can be calculated from the expression,

$$U_j^\alpha(z) = \sum_{t=1}^{T} \pi_t(z) U_j^t(z) = \sum_{t=1}^{T} \pi_t(z) \widetilde{V}_j\big(\tilde{g}_t^\alpha(z)\big). \tag{4}$$

We have implicitly assumed that the preference profile of the party principals is parametrized by a vector $(y, \alpha) \in Z^p \times X$, where y represents the vector of ideal points of party principals, and $\alpha \in X$. We shall assume X is a compact set of parameters in $\Re^p$. Given the game form $\tilde{g}$, and the parameter α, we show the dependence of the profile map by writing $U(\tilde{g}, y, \alpha) : Z^p \to \Re^p$. The components of this function are well-defined by equation (4). Assumptions 1–4 can be restated as assumption 5.

Assumption 5. For each parameter $(y, \alpha) \in Z^p \times X$, the game form $\tilde{g}$ induces a C^1-differentiable profile map $U(\tilde{g}, y, \alpha) : Z^p \to \Re^p$ on the joint strategy space Z^p.

Up to this point we have assumed that the principals believe that bargaining between the party leaders within the decision structure, $\mathcal{D}_t$, can be described by the lottery $\tilde{g}_t^\alpha(z)$. Under some conditions this lottery has support restricted to a particular subset of W. Previous work [31] has proposed a solution notion, called the "heart", conceptually similar to the so-called "uncovered set". Given the profile $q^\alpha(z)$ of leaders' preferences, and the family $\mathcal{D}_t$ of decisive coalitions, then the heart, denoted $\mathcal{H}_t^\alpha(z)$, will be a subset of W. Now let $\widetilde{\mathcal{H}}_t^\alpha(z)$ denote the set of all probability measures on W with support $\mathcal{H}_t^\alpha(z)$, and let $2(\widetilde{W})$ be the power set of $\widetilde{W}$. Section 3 refers to previous results showing that the correspondence $\widetilde{\mathcal{H}}_t^\alpha : Z^p \to 2(\widetilde{W})$ admits a smooth selection $g : Z^p \to \widetilde{W}$ (with respect to the weak topology on $\widetilde{W}$). That is, there exists a smooth function $g : Z^p \to \widetilde{W}$ such that $g(z) \in \widetilde{\mathcal{H}}_t^\alpha(z)$, for all $z \in Z^p$.

Definition 1. The smooth game form $\tilde{g}$, which specifies $\tilde{g}(\alpha) = \{g_t^\alpha, \pi_t\}_T$ at α, is *heart compatible* iff each component $\tilde{g}_t^\alpha : Z^p \to \widetilde{W}$ is a smooth selection of the heart correspondence $\widetilde{\mathcal{H}}_t^\alpha : Z^p \to 2(\widetilde{W})$.

We now define the notion of a *local pure strategy Nash equilibrium* (LNE) for the profile map $U : Z^p \to \Re^p$ given by $U(z) = (U_1(z), \dots, U_p(z))$.

Definition 2.

(i) A LNE (for the party system P and profile map $U : Z^p \to \Re^p$) is a vector $z^* \in Z^p$ such that, for each $i \in P$, there is a neighborhood K_i of z_i^* in Z, with the property that

$$U_i\big(z_1^*, \ldots, z_i, z_{i+1}^*, \ldots, z_p^*\big) > U_i\big(z_1^*, \ldots, z_i^*, z_{i+1}^*, \ldots, z_p^*\big)$$

for no $z_i \in K_i$.

(ii) A LNE $z^* \in Z^p$ is *locally isolated* iff there is a neighborhood K^* of z^* in Z^p which contains no LNE other than z^*.

(iii) A *Global Nash Equilibrium* (GNE) is a LNE with the additional requirement that each K_i is in fact Z.

It is well known that various assumptions on continuity and convexity of "reaction functions" induced from the profile can guarantee existence of a GNE. Since, convexity will generally fail, it is common to focus on MSNE, using randomization across pure strategies. For empirical reasons, given in section 4, we emphasize LNE instead. Simulation techniques, based on estimating the electoral function π, does permit calculation of LNE. However, our empirical analysis suggests that GNE generally do not exist.

Before stating our main theorem, we need to introduce a technical property on the game form $\tilde{g}$. We have assumed that the policy space Z is some compact subset of $\Re^w$. In discussing the electoral map π in section 4, we implicitly assume that π is determined by voter responses, where voters are characterized by ideal points in some compact convex subset Z' of Z. Thus it makes sense to impose the additional criterion that if party i chooses a leader with policy position in the set $Z \setminus Z'$, then the electoral support of the party will be zero. We assume therefore that no party will consider a leader with a position outside Z'. When the game form $\tilde{g}$ satisfies this assumption for some such subset Z', then we shall say it is *bounded* in Z'.

Theorem 1. If the game form $\tilde{g}$ is bounded in Z' and satisfies assumption 5, then there is an open dense set $K_g \subset Z^p \times X$ such that the induced profile $U(\tilde{g}, y, \alpha) : Z^p \to \Re^p$ exhibits a locally isolated LNE for all $(y, \alpha) \in K_g$.

Theorem 2. Theorem 1 also holds when the game form $\tilde{g}$ is further required to be heart-compatible.

The proof of these results follow in the next section. The final section offers some empirical justification for the model that we have proposed.

3. Formal definitions and proof of theorems

A *preference* Q on a set, or space, W is a correspondence $Q : W \to 2(W)$. Again, $2(W)$ stands for the power set of W, namely the family of all subsets of W (including the empty set ϕ). Q is *strict* if $y \in Q(x)$ implies not ($x \in Q(y)$). For convenience we say W is a *Fan space* if it is a compact convex subset of a linear topological space of finite dimension, and has a smooth boundary.

Definition 3.

(i) Let $Q : W \to 2(W)$ be a strict preference correspondence on the space W. The *core* of Q is $E(Q) = \{x \in W\colon\ Q(x) = \phi\}$.

(ii) The covering correspondence, $\overline{Q}$ of Q is defined by $y \in \overline{Q}(x)$ iff $y \in Q(x)$ and $Q(y) \subset Q(x)$. Say y *covers* x.
The *uncovered set*, $\overline{E}(Q)$ of Q, is $\overline{E}(Q) \equiv E(\overline{Q}) = \{x \in W\colon\ \overline{Q}(x) = \phi\}$.

(iii) If W is a topological space, then $x \in W$ is *locally covered* (under Q) iff for any neighborhood K of x in W, there exists $y \in K$ such that $y \in Q(x)$ and $K \cap Q(y) \subset K \cap Q(x)$. If x is not locally covered, then write $\widehat{Q}(x) = \phi$.

(iv) The *heart* of Q, written $\mathcal{H}(Q)$, is defined by

$$\mathcal{H}(Q) = \big\{x \in W\colon\ \widehat{Q}(x) = \phi\big\}.$$

Now let $Q^*(W)^P$ stand for all "smooth" convex preference profiles for the society P. Thus $q \in Q^*(W)^P$ means $q = (q_1, \dots, q_p)$ where each q_j is a convex preference, induced from a C^1-differentiable "quasi-concave" utility function, V_j'.

Definition 4. Let $\mathcal{D}$ be a fixed voting rule (namely a family of decisive coalitions). Let $q \in Q^*(W)^P$ be a smooth preference profile on the Fan space, W. Define $\sigma_{\mathcal{D}}(q) = \bigcup_{M\in\mathcal{D}}\{\bigcap_{j\in M} q_j\} : W \to 2(W)$ to be the preference correspondence induced by $\mathcal{D}$ at q.

The *core* of $\mathcal{D}$ at q, written $E_{\mathcal{D}}(q)$, is $E(\sigma_{\mathcal{D}}(q))$.

The *heart* of $\mathcal{D}$ at q, written $\mathcal{H}_{\mathcal{D}}(q)$, is defined to be $\mathcal{H}(\sigma_{\mathcal{D}}(q))$. The *uncovered set* of $\mathcal{D}$ at q, written $\overline{E}_{\mathcal{D}}(q)$, is $\overline{E}(\sigma_{\mathcal{D}}(q))$.

The *Pareto set* of the profile q is $E_P(q) = E(\sigma_P(q))$ where $\sigma_P(q) : \{\bigcap_{j\in P} q_j\} : W \to 2(W)$ is the Pareto, or strict unanimity, preference correspondence.

A correspondence $Q : W \to Y$ is upper hemi-continuous (uhc) with respect to topologies on W, Y iff for any open $K \subset Y$ the set $\{x \in W\colon\ Q(x) \subset K\}$ is open in W.

A correspondence $Q : W \to Y$ is lower hemi-continuous (lhc) with respect to topologies on W, Y iff for any open $K \subset Y$ the set $\{x \in W\colon\ Q(x) \cap K \neq \phi\}$ is open in W. A continuous selection g for Q is a function $g : W \to Y$, continuous with respect to the topologies on W, Y such that $g(x) \in Q(x)\ \forall x \in W$, whenever $Q(x) \neq \phi$.

The uncovered set of a voting rule, $\mathcal{D}$, at a profile q, has been proposed as a solution concept for "spatial" voting games when the core is empty (see [6,11,23,26]). Under certain conditions the uncovered set and the core coincide when the latter is nonempty [7]. Using the Hausdorff topology on preference profiles, Banks et al. [7] have shown that the uncovered set correspondence is uhc "near" a core profile. Thus the uncovered set "strongly converges" to the core. The heart correspondence has the "weaker" property of lhc, when its domain is $Q^*(W)^P$.

More precisely, Schofield [31,33–35] has shown that the heart is nonempty, Paretian and a lhc correspondence, with respect to a C^1-topology [32] on $Q^*(W)^P$. This C^1-topology utilizes gradient information of the representative utility functions $\{V'_j\}$.

The uncovered set, heart and Pareto set can, of course, be defined for non-convex preference. For a convex preference profile, the heart will contain the uncovered set. This property need not hold, however, for a non-convex preference profile. (See [35, figure 2].)

The following three propositions summarize the technical properties of the heart correspondence.

Proposition 1. Let W be a Fan space, and $\mathcal{D}$ any voting rule. Then $\mathcal{H}_{\mathcal{D}}: Q^*(W)^P \to 2(W)$ is lhc (with respect to the C^1-topology on the domain, and Euclidean topology on W). Moreover, for any $q \in Q^*(W)^P$, $\mathcal{H}_{\mathcal{D}}(q)$ is closed, nonempty and is a subset of the Pareto set $E_P(q)$. If $E_{\mathcal{D}}(q) \neq \phi$ and $x \in E_{\mathcal{D}}(q)$, then, for any sequence (q_s) of profiles in $Q^*(W)^P$ which converges to q, in the C^1-topology, there exists a sequence (z_s) in W which converges to x, in the Euclidean topology on W, such that $z_s \in \mathcal{H}_{\mathcal{D}}(q_s)$ for all z_s in the sequence. (See [35, theorem 2].)

This latter property we term "the weak convergence of the heart to the core".

To use the results to model coalition bargaining, we assume that the choice of the *leader* (or *agent*) for party j determines the declaration z_j of the party. We assumed in equation (3) that the leader has a utility function of the form

$$V'_j(\chi, \delta_j) = -A_j(z_j, \chi) + \alpha_j \delta_j.$$

This generates a smooth, strictly convex preference correspondence for the leader of party j of the form $q_j^{\alpha_j}(z_j): W \to 2(W)$, where $W \equiv Z \times \Delta_P$. We denote the induced leader profile by $q^\alpha(z)$. Clearly $q^\alpha(z) \in Q^*(W)^P$.

The Pareto set in W is the unanimity choice of these preferences. For a fixed voting rule $\mathcal{D}_t$, using definition 4, we can define the *heart* of the voting rule on the space W as $\mathcal{H}_{\mathcal{D}_t}(q^\alpha(z))$. However, since the profile $q^\alpha(z)$ is fully specified by the vectors α and z, we may write this object as $\mathcal{H}_t^\alpha(z)$. Proposition 1 can then be used to show that $\mathcal{H}_t^\alpha$ is lhc and weakly converges to the voting core $E_t^\alpha(z) = E_{\mathcal{D}_t}(q^\alpha(z))$, if the core is nonempty. Lower hemi-continuity of $\mathcal{H}_t^\alpha$ allows use of Michael's Selection Theorem [25] to show existence of a selection, g_t^α. Moreover, since $\mathcal{H}_t^\alpha$ is lower hemi-continuous with respect to a C^1-topology [33] on smooth utility profiles, the selection g_t^α can be chosen to be C^1-differentiable.

Proposition 2. Let Z be a compact convex subset of $\Re^w$, endowed with the Euclidean topology, and let Z^p be the product space. Then for any voting rule, $\mathcal{D}_t$, and any parameter α, $\mathcal{H}_t^\alpha : Z^p \to 2(W)$ is lhc.

For each $z \in Z^p$, $\mathcal{H}_t^\alpha(z)$ is a closed, nonempty subset of the Pareto set in W. Moreover, $\mathcal{H}_t^\alpha$ admits a C^1-differentiable Paretian selection g_t^α, where $g_t^\alpha : Z^p \to W$. (See [35, proposition 5].)

Let $\widetilde{W}$ be the set of all measures on W, endowed with the weak topology, and let $\widetilde{\mathcal{H}}_t^\alpha : Z^p \to 2(\widetilde{W})$. As before, $\widetilde{\mathcal{H}}_t^\alpha(z)$ denotes the set of probability measures, with support $\mathcal{H}_t^\alpha(z)$, and the induced weak topology. Then lhc of $\mathcal{H}_t^\alpha$ implies lhc of $\widetilde{\mathcal{H}}_t^\alpha$ [34, corollary 5].

Proposition 3. For a fixed voting rule, $\mathcal{D}_t$, there exists a smooth selection $\tilde{g}_t^\alpha : Z^p \to \widetilde{W}$ of $\widetilde{\mathcal{H}}_t^\alpha$.

Proposition 3 follows directly from proposition 2 by observing that if $g_t^\alpha : Z^p \to W$ is a C^1-differentiable selection of $\mathcal{H}_t^\alpha$, then it can be regarded as a function $g_t^\alpha : Z^p \to \widetilde{W}$. Clearly $g_t^\alpha(z) \in W$ can be regarded as a measure on $\widetilde{W}$ with all probability weight on the point $g_t^\alpha(z)$. Since $\mathcal{H}_t^\alpha(z) \subset \widetilde{\mathcal{H}}_t^\alpha(z)$, $g_t^\alpha \equiv \tilde{g}_t^\alpha$ gives a C^1-differentiable selection of $\widetilde{\mathcal{H}}_t^\alpha$.

Proposition 3 shows that it is, in principle, possible to find a game form which is "heart compatible".

We now show that LNE exist and are "generically" locally isolated. By "generically", we mean for almost all values of $\alpha \in X$ and ideal points $y \in Z^p$ of the party principals. More formally, let $\mathcal{G} = \{U\colon Z^p \to \Re^p\}$ be the topological space of all C^1-differentiable utility profiles induced by bounded game forms which satisfy assumption 5. A generic property of $\mathcal{G}$ is a property of all profiles belonging to an open-dense subset of $\mathcal{G}$ (when endowed with the C^1-topology). See [17,19] for the formal definition of "genericity".

We use the term *Critical Nash Equilibrium* (*CNE*) at a profile U for a vector $z^* \in Z^p$ which satisfies the condition $\mathrm{d}U_j/\mathrm{d}z_j = 0$, for all j. Let T_j^α be the subset of Z^p where this first order condition for U_j is satisfied.

Let $\mathcal{K}$ be the subset of $\mathcal{G}$ such that each $U \in \mathcal{K}$ has at least one LNE which is locally isolated.

Theorem 3. The set $\mathcal{K} \subset \mathcal{G}$ is open-dense in the C^1-topology on $\mathcal{G}$.

Proof. Consider a typical profile U in $\mathcal{G}$. For each $z \in Z^p$, z_j^* is chosen to satisfy the first order condition $\mathrm{d}U_j/\mathrm{d}z_j = 0$. By the inverse function theorem, T_j^α, is generically a smooth manifold of dimension $(p-1)\dim(Z)$. By the Thom Transversality theorem, the intersection $\bigcap_{j\in P} T_j^\alpha$ is generically of codimension $p\dim(Z)$ in Z^p. (See [17,19] for the formal statement of the theorem and [30] for applications.) However, Z^p has dimension p dim (Z). Since the CNE $\equiv \bigcap_{j\in P} T_j^\alpha$, this shows the CNE is generically of dimension 0. That is, it consists of locally isolated points. We may now construct a gradient field μ on Z^p whose zeros consist precisely of the CNE (see [32] for this construction). Since Z is homeomorphic to the ball, it has Euler characteristic 1. Since every game form is assumed to be bounded, the field μ points inward on the boundary of Z^p. The Morse inequalities [14,27] imply that there must be at least one critical point of μ whose index is maximal. This point corresponds to a locally isolated LNE. Since Z is compact, $\mathcal{K}$ is open dense in $\mathcal{G}$. □

Proof of theorem 1. The game form $\tilde{g}$ generates a map $g^* : Z^p \times X \to \mathcal{G}$ by $g^*(y, \alpha) = U(\tilde{g}, y, \alpha)$. Clearly this map is continuous, and the inverse K_g of the open dense set $\mathcal{K}$ in $\mathcal{G}$, is also open dense. Thus there exists an open dense set K_g in $Z^p \times X$, such that $U(\tilde{g}, x, \alpha)$ has locally isolated and nonempty LNE, whenever $(x, \alpha) \in K_g$. □

Theorem 2 follows immediately by applying the proof technique of theorem 3 to the subset of $\mathcal{G}$ generated by heart compatible game forms.

4. Empirical analysis

In the model constructed in section 2 we have assumed that the electoral map $\pi : Z^p \to \Delta_T$ is smooth. Since we also assume that the game form $\tilde{g}_t^\alpha$ is functionally dependent on $\mathcal{D}_t$, this implicitly means that bargaining between party leaders is not dependent on seat shares *per se*, but only on coalition structures. Most models of elections focus on vote (or perhaps seat) shares, so it is necessary to offer a method of estimating π. Since π represents beliefs of party principals, we propose a proxy, Φ, which is based on electoral sampling.

At the election time, let $N = \{1, \ldots, n\}$ be some sample. Each voter i when presented with a vector $z \in Z^p$ of leadership positions (or manifestos) makes a choice based on a p-vector of utility functions given by equation (1).

In the empirical work we assumed $A_j(x_i, z_j) = \beta \|z_j - x_i\|^2$. As before, x_i is voter i's preferred policy point. The probability, $\rho_{ij}(z)$, that voter i chooses party j is $\text{Prob}[\bar{u}_{ij}(z) > \bar{u}_{ik}(z)$ for all $k \neq j]$.

Information on party positions as well as voter positions and characteristics can in fact be used to estimate a multinomial logit (MNL) model of the election. (See [29,37] for a discussion of this procedure.) For example, figures 1 and 2 show the estimated sample distribution of the voter preferred points for the 1992 and 1996 elections in Israel (using a sample size of approximately 1000). The samples were collected by Arian and Shamir [2,3], and factor analysis was used to derive the two-dimensional policy space, Z. Each member of the sample could then be represented by an estimated bliss point in Z. The manifestos of the parties were studied, using the survey questionnaires to derive party positions, and thus an estimate for the vector $z \in Z^p$. As in the discussion of the model above, we assume the party positions were, in fact, those of the party leaders. See [39] for an earlier study of the 1992 election, including details on the questionnaires. The shaded regions in figures 1 and 2 represent the 95, 75, 50 and 10 percent highest density regions of voter bliss points.

The empirical model thus estimates for each sample voter, i, the utility p-vector, $\bar{u}_i(z)$ (as in equation (1)) and a probability p-vector, $\rho_i(z) \in \Delta_p$. Summation and normalization then gives a stochastic vote variable $\Phi(z)$ whose components are the random variables characterizing the vote shares of the p-parties. Thus $\Phi : Z^p \to \tilde{\Delta}_p$, where $\tilde{\Delta}_p$

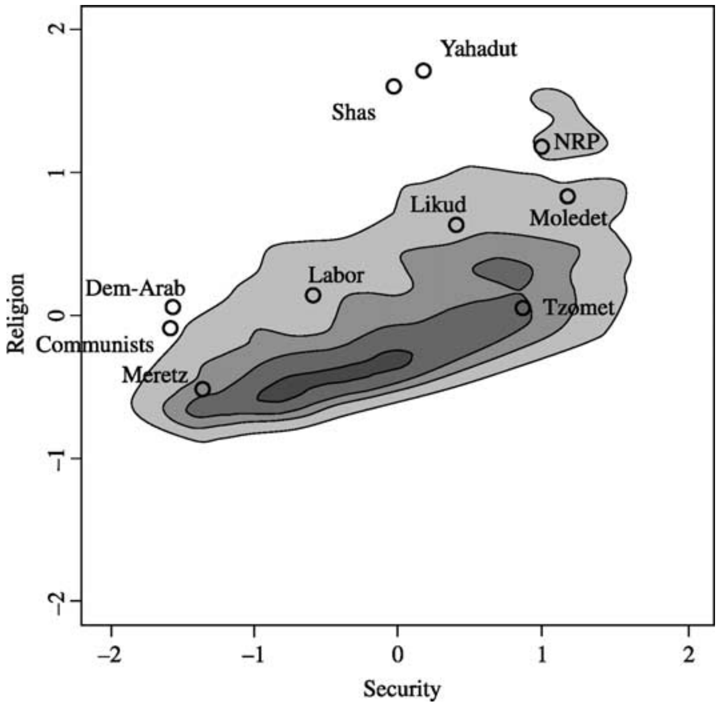

Figure 1. Israeli party positions in 1992, showing the highest density plot of the voter sample distribution at the 95%, 75%, 50% and 10% levels.

is the space of probability measures on Δ_p. The estimation procedure is devised to fit the sample data, so that the expectation $\mathcal{E}(\Phi)(z) \in \Delta_p$ matches the sample vote shares. However, because the MNL model is based on Gaussian assumptions on the errors, both the expectation and variance operators $\mathcal{E}(\Phi)$ and var (Φ) will be smooth functions on Z^p.

If we assume that the electoral system is purely proportional, then vote shares will translate directly into seat shares. Thus the stochastic vote map $\Phi : Z^p \to \tilde{\Delta}_p$ gives information on the electoral map $\pi : Z^p \to \Delta_T$. The electoral system translates vote shares into seat shares. Let $\Psi : Z^p \to \tilde{\Delta}_p$ be the estimated seat share map. The computation of π involves various combinatorial expressions. If we let $\Psi_j(z)$ be the jth component of $\Psi(z)$, and assume that each coalition, M, in $\mathcal{D}_t$, say, is characterized by the expression $\sum_{j\in M} \Psi_j(z) > \frac{1}{2}$, then $\pi_t(z)$ can be computed from the compound probability that $\sum_{j\in M} \Psi_j(z) > \frac{1}{2}$, for each $M \in \mathcal{D}_t$ and $\sum_{j\in M} \Psi_j(z) \leqslant \frac{1}{2}$, for each $M \notin \mathcal{D}_t$. The smoothness of the expectation and variance functions then implies that π is C^1-differentiable. The typical probabilistic vote model [5,21], for empirical applications, can be interpreted to mean that each party, j, chooses a position so as to maximize its expected vote share (namely the jth component of $\mathcal{E}(\Phi)(z)$). We therefore studied the game given by a profile map $\mathcal{E}(\Phi) : Z^p \to \Re^p$, and examined existence of LNE under the assumption of vote share maximization.

We tested the assumption that the political game had this form by using our fit-

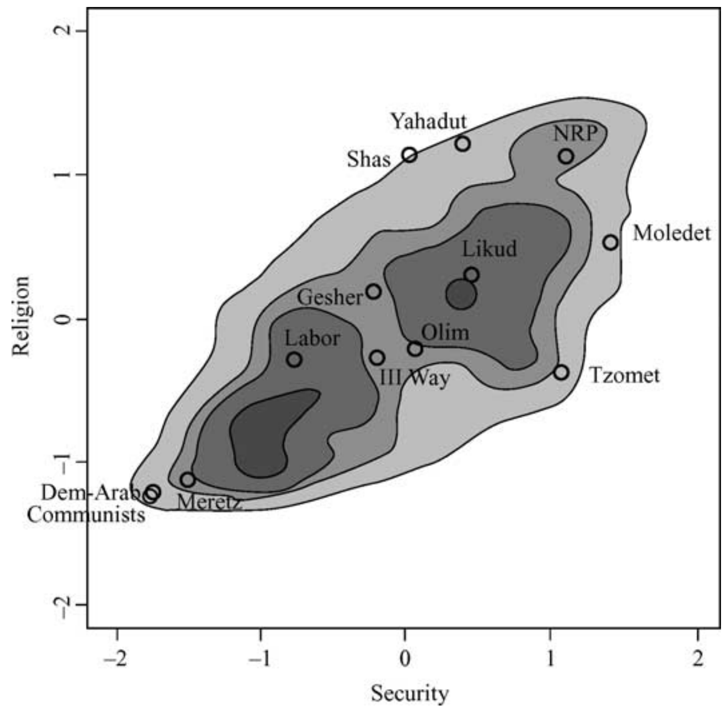

Figure 2. Israeli party positions in 1996, showing the highest density plot of the voter sample distribution at the 95%, 75%, 50% and 10% levels.

ted equation (1) for the voter sample, and by examining the relationship between the vector z and the estimated vote share function $\mathcal{E}(\Phi)(z)$. Simulation based on "mountain climbing" algorithms showed that there were multiple LNE. Moreover, in contrast to the theoretical results of Lin et al. [21] the party positions in LNE were neither similar nor centrist.

Typical locations for the Labor and Likud LNE positions in 1992 and 1996 were close to the positions of these two parties in figures 1 and 2. However, the LNE position for Shas in 1996 was close to the position (1.1, 1.1), whereas in 1992 it was close to (0.5, 0.7). This estimation suggests that Shas was quite far in 1992 from its vote maximizing position. While it gained both votes and seats in 1996 by moving very slightly towards the electoral center, it could have, according to our estimation, gained further support by changing its leader's position, to an even more centrist position.

We suggest that the parties such as Shas may in fact be modifying their expected vote, and seat shares, by acting to change the constant term, λ, in the voter equation (1).

In our analysis, the estimated constant term, λ, for Shas increased from -6.16 in 1988 to -2.96 in 1996. Clearly a significant estimated negative term will tend to reduce average vote share. The fact that this negative effect became less pronounced parallels the increase in the vote share for Shas.

This phenomenon suggests that parties attempt to obtain activist support from voters, and these contributions directly affect the constant terms in the voter equations. Changes in such activity then affect voter response and hence expected vote.

An alternative possibility is that Shas was able to use its success in coalition bargaining to further increase activist support. For example, after the 1996 and 1999 elections, Shas was pivotal between governments led either by Labor or Likud. Under the leadership of Netanyahu, a Likud led government after 1996 was willing to offer both policy and non-policy rewards to Shas to retain its support. By this means, Shas could show its success to its supporters. Further support from them would then increase voter support at the next election. More generally this inference suggests that voter choice is governed by an expression of the form

$$\bar{u}_i(z) = \lambda(z) + \varepsilon - A(x_i, z) + B\gamma_i. \tag{5}$$

Here λ is no longer a constant p-vector but a function $\lambda : Z^p \to \Re^p$. If λ is assumed to be C^1-differentiable, then the probability functions $\rho_i : Z^p \to \Delta_p$ and $\Phi : Z^p \to \tilde{\Delta}_p$ will be C^1-differentiable and smooth, respectively. Thus the induced electoral map $\pi : Z^p \to \Delta_T$ will still be C^1-differentiable.

The model proposed in this paper emphasizes the fact that party principals are concerned with policy and perquisites. It may of course follow that they have an interest in acting so as to maximize expected vote share, but this will depend on their beliefs about the nature of coalition bargaining. To pursue this phenomenon, consider the post election bargaining after the 1996 election. For purposes of exposition, let us ignore the effect of perquisites by assuming $\alpha \equiv 0$. It should be clear from table 1 and figure 2 that a coalition of the right, embracing Likud, Shas, and other centrist and religious parties controlled a majority (namely 68 seats out of 120). However, Labor, Shas and Meretz, and the small Arab parties controlled 62 seats, while Likud and Labor controlled 64. Let us denote this decisive coalition structure by $\mathcal{D}_1$. Since we assume that the policy preferences of the leaders are Euclidean (by equation (3)) the leadership profile $q^0(z)$ generates a heart $\mathcal{H}_1^0(z)$ which is the convex hull of the positions of the leaders of Labor, Olim and Shas. (We assume that the vector z of party leader positions is given by the party positions of figure 2.)

Note to table 1 and the figures

There are some slight differences between table 1 and the electoral results for 1988–1996 in Schofield, Sened and Nixon [39].

In the above table, 9 seats are allocated to "others" in 1988. In fact these seats were distributed among three small parties (CRM, Mapam and PLP). The two parties denoted Communist and Democratic Arab in the above table are termed Hadash (HS) and Daroushe (ADL) respectively in Schofield et al. [39]. The NRP above is termed Mafdal in Schofield et al. [39]. Yahadut above, comprises two parties: Aguda (which had 5 seats in 1988 and 4 in 1992) and Degel Hatora (which had 2 in 1988).

In 1996, Olim gained 7 seats. This party was called Israel Ba'aliya (or "Russian" party) in Schofield et al. [39].

Table 1
Elections in Israel.

	Party	Knesset seats			
		1988	1992	1996	1999
Left	Labor	39	44	34	28
	Meretz	–	12	9	10
	Shinu	2	–	–	6
	Others	9	–	–	–
	Democrat Arab	1	2	4	5
	Communists	4	3	5	3
	Balad	–	–	–	2
Subtotal		*55*	*61*	*52*	*54*
Center	Olim	–	–	7	6
	Third Way	–	–	4	–
	Center	–	–	–	6
Right	Likud	40	32	30	19
	Gesher	–	–	2	–
	Tzomet	2	8	–	–
	Yisrael Beiteinu	–	–	–	4
Subtotal		*42*	*40*	*43*	*35*
Religious	Shas	6	6	10	17
	Yahadut	7	4	4	5
	NRP	5	6	9	5
	Moledet	2	3	2	4
	Techiya	3	–	–	–
Subtotal		*23*	*19*	*25*	*31*
TOTAL		*120*	*120*	*120*	*120*

All of the above are alternative names that are used interchangeably in Israel and in the literature.

Slight differences in party positioning for figures 1 and 2 in contrast to figures 2 and 3 in Schofield et al. [39] may be noticed. This is due to a more refined factor analysis procedure adopted for the analysis presented here.

There are a number of post election bargaining models, for example, [4,8]. In general these require consideration of every coalition in $\mathcal{D}_1$. Since there were 12 parties in the Knesset in 1996, the coalition structure was extremely complex. However, a much simpler framework for coalition bargaining is to assume that the policy outcome belongs to the heart. More generally, a reasonable pre-election belief of the parties is that the lottery $\tilde{g}_1^0(z)$, that results from bargaining, under $\mathcal{D}_1$, is the uniform distribution across the set $\mathcal{H}_1^0(z)$. It should be evident that the function $\tilde{g}_1^0 : Z^p \to \widetilde{Z}$ is both smooth, and a selection for the heart correspondence $\widetilde{\mathcal{H}}_1^0 : Z^p \to 2(\widetilde{Z})$.

Schofield and Parks [38] essentially studied existence of LNE using simulation techniques in the situation with $p = 3$, and a single decisive structure similar to $\mathcal{D}_1$. Depending on the location of the bliss points of the principals, they showed that LNE could generate divergent party positions. When the effect of perquisites (assuming both α_i and

$\delta_i \neq 0$) were considered, then the divergence effect became less marked. However, GNE in general did not exist. Since the formal model proposed here is a generalization of this earlier 3-party version, GNE cannot be expected to occur.

The divergence effect found in the 3-party model will be lessened if the probability of a core has to be considered. To illustrate this phenomenon, consider the 1992 election. The parties on the right, including both Likud and Shas, controlled 59 seats after the election. Assuming again that $\alpha \equiv 0$, using $\mathcal{D}_2$ to denote the coalition structure, and $q^0(z)$ to denote the leader profile (as given in figure 1) it is easy to show that the core $E_2^0(z)$, in Z, is nonempty. In fact, it is located at the position of the Labor leader, in figure 1. In this case the heart $\mathcal{H}_2^0(z)$ and the core are identical. Empirical work by Laver and Schofield [20] suggests that the Labor party could form a minority government. The formal bargaining model of Banks and Duggan [4] supports this conclusion, since their model essentially implies that a party, such as Labor, in the core position, would control policy in this situation. In fact, Labor under Rabin did form a minority government, and as a consequence was able to initiate the peace accords with the PLO. It is natural therefore to represent the pre-election beliefs over coalition bargaining as follows: If the decisive structure $\mathcal{D}_2$ occurs, and the Labor position is at the nonempty core position, $E_2^0(z)$, then Labor will control government. Since this is attractive for Labor and unattractive for Likud, Labor should rationally adopt a position in order to maximize the probability, π_2, that $\mathcal{D}_2$ occurs. In contrast Likud should maximize the probability, π_1, that $\mathcal{D}_1$ occurs. A simpler objective function for these two parties is therefore to maximize expected vote share. This does not necessarily hold true for Shas. If the principal of Shas believes that the probability π_1 is significantly higher than π_2, and assuming the lottery $\tilde{g}_1^0(z)$ will be a uniform distribution over the heart, then it should choose a leader whose (possibly "divergent") policy position will lead to effective bargaining with Labor or Likud.

Although Banks and Duggan [5] and Lin et al. [21] obtain conditions under which parties converge to an identical position (under expected vote maximization) this phenomenon does not occur in the more general model proposed here. Even though Likud and Labor may have acted like vote maximizing parties, convergence did not occur. Since polling data (which we have proxied by Φ) would lead Shas to infer that $\mathcal{D}_2$ would be the most probable coalition structure at the 1996 election, it should rationally position itself to take advantage of its pivotal position. Assuming that the leader positions are LNE implies that vote maximizing positions for Labor and Likud are dependent on the Shas position, and are not convergent.

The notion of the post election heart that we have proposed offers additional insight into the post-election bargaining game. In 1996, Netanyahu (of Likud) won a separate prime minsterial election against Peres (of Labor). However, to create a winning governmental coalition, Netanyahu had to attract both Shas and religious parties. Increasing disagreements on the security dimension (over policy with regard to the PLO) led to government collapse. Barak, the new leader of Labor won the 1999 election, but faced the same problem of coalition maintenance.

The current government is led by Sharon, of Likud, and incorporates the parties “bounding” the heart, namely Shas, and Labor.

Although we have postulated the existence of LNE in pre-election party maneuvers, this does not of course entail “equilibrium” in any sense in post-election coalition maintenance. However, the creation of a multiparty coalition based on the heart by Sharon suggests that this was an attempt by him to overcome political disagreements between the Israeli parties in the presence of the rapidly increasing conflict with the Palestinian Authority.

5. Conclusion

The model proposed here is devised to be general enough to accommodate complex jockeying between parties as they consider both electoral consequences and the effects of leader positions on post election bargaining. Simplified versions of the model, when applied to complex political situations such as Israel, give a number of possible explanations why parties appear not to converge to the electoral center. This phenomenon has been known in a general sense [13,15] for many years. More recent empirical work [1,18,22,36] has suggested that parties, both in proportional European systems, as well as in the United States and Britain, remain distinct from one another in their policy choices.

Because of the complexity of the underlying model, quasi-concavity of the utility functions cannot be expected. Instead the focus has been on local optimization. This seems an appropriate methodology to utilize since parties will tend to be restricted in their choice of party leaders and supporters. While the electoral component of the model is compatible with empirical work to date, further more refined estimation of the beliefs of the political elite over post-election bargaining can be used to make more robust inferences about the nature of party positioning in fragmented electoral environments.

Acknowledgments

Empirical research discussed in this paper was done in collaboration with Andrew Martin, David Nixon, Dganit Ofek, Robert Parks, Kevin Quinn and Andrew Whitford, with support from NSF Grant SBR 98-18582. We appreciate the helpful comments of Jeff Banks, John Duggan, and Roger Myerson. Versions of the paper were presented at the 3rd International Conference on Game Theory, Barcelona, June 1998; at the 12th World Congress of the International Economic Association, Buenos Aires, 1999; and at the Meeting of the Society for Public Economic Theory, University of Warwick, July 2000, and at the Meeting of the Society for the Advancement of Economic Theory, Ischia, Italy, July 2001.

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