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Modeling the effect of campaign advertising on US presidential elections when differences across states matter

Maria Gallego [a], Norman Schofield [b,*]

[a] Department of Economics, Wilfrid Laurier University, 75 University Avenue West, Waterloo, Canada N2L 3C5
[b] Department of Political Science, Washington University in Saint Louis, Campus Box 1027, One Brookings Drive, St. Louis, MO 63130-4899, United States

HIGHLIGHTS

- Policy and advertising campaigns when differences in across states matter.
- Candidates announce policy and advertising campaign.
- Voters' preferences: policy, ad messages and valences.
- Candidates give maximal weight to undecided voters & swing states in campaign.
- Weights vary across policy and ad campaigns and across candidates.

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ABSTRACT

We provide a stochastic electoral model of the US Presidential election where candidates take differences across states into account when developing their policy platforms and advertising campaigns. Candidates understand the political and economic differences that exist across states and voters care about candidates' policies relative to their ideals, about the frequency of candidates' advertising messages relative to their ideal message frequency, their *campaign tolerance level*, and vote taking into account their perceptions of candidates' traits and competencies with their vote also depending on their sociodemographic characteristics. In the local Nash equilibrium, candidates give maximal weight to undecided voters and swing states and little weight to committed voters and states. These endogenous weights pin down candidates' campaign and depend on the probability with which voters choose each candidate which depends on candidates' policies and advertising campaigns. Weights vary across candidates' policy and ad campaigns, reflecting the importance voters in each state give to the two dimensions and the variation in voters' preferences across states.

1. Introduction

This paper studies the effect of campaign advertising on US presidential elections. Studying campaign advertising has become particularly important since the US Supreme Court's 2010 decision in *Citizens United vs. Federal Election Commission* removing limits on campaign contributions. Contributions substantially increased after this ruling, giving candidates the funds necessary to run extensive advertising campaigns.[1]

We develop a model of US Presidential elections where candidates use two instruments in their electoral campaigns – policy and advertising – to address different aspects of the election. Policies are those candidates want to implement if elected with the advertising component aimed at giving voters a further impetus to vote for candidates. These two instruments exert then a differential impact on voters choices in the election. We develop a model in which candidates use these two electoral campaign instruments in their bid for office and allow voters to have preferences over candidates' policies and campaign advertising.

The collection of articles included in Hendricks and Kaid's (2011) edited volume *Techno Politics in Presidential Campaigning: New Voices, New Technologies and New Voters*, argue that the 2008 US presidential election was a landmark campaign because it elected the first African–American President and brought about dramatic changes in the way presidential campaigns would be conducted from 2008 onwards. We argue that the emergence

* Corresponding author.
E-mail addresses: mgallego@wlu.ca (M. Gallego), schofield.norman@gmail.com (N. Schofield).

1 Ansolabehere et al. (2003) estimate that the marginal impact of $100,000 spent in a House race, ceteris paribus, is about 1% gain in vote. Thus, highlighting the powerful influence of ad campaigns on elections.

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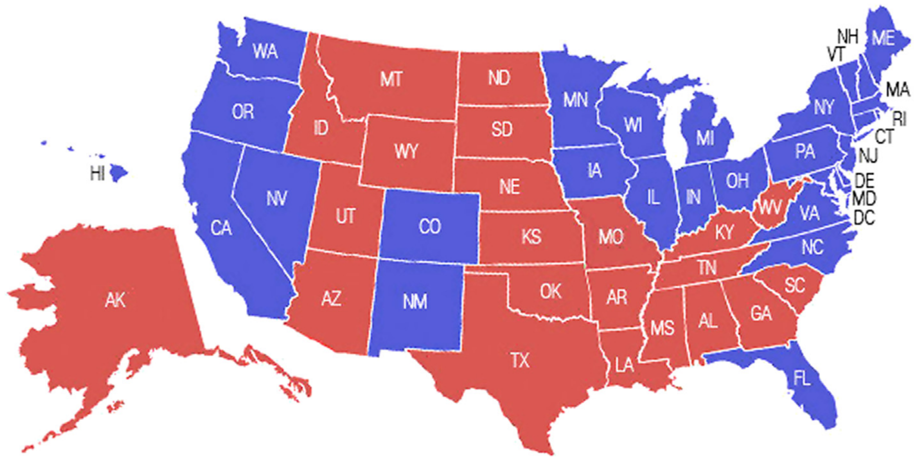

Fig. 1. The 2008 US Presidential Election map. Obama won the blue states, McCain those in red. Retrieved from NPR, http://www.npr.org/blogs/itsallpolitics/2012/11/01/163632378/a-campaign-map-morphed-by-money. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

of smart phones and social media[2] coupled with large data sets on voters' personal information[3] and public and private pre-election polls has had a profound effect on political campaign advertising and on US Presidential elections as new media and new technologies give voice to new candidates and voters. The congruence of these factors has revolutionized the way electoral campaigns are conducted by allowing candidates to tailor their campaign message to voters taking their sociodemographic characteristics and political preferences into account. Campaign strategists can now identify candidates' core supporters and undecided voters.

By directly communicating with voters, candidates have greater control over their message than when using mass media (TV and radio).[4] We assume candidates directly communicate with voters and allow voters to have preferences over the frequency with which candidates contact them, i.e., voters are also identified by their *campaign tolerance level*.[5]

Candidates understand the differences that exist across states[6] as illustrated in the following cartograms of the 2008 US Presidential election. Fig. 1, the electoral map of the election,[7] would lead us conclude that McCain won the election as there are more red than blue states, yet it is Obama who won. This visual distortion is due to the fact that the US map does not reflect the distribution of the population by state. The cartogram is indicative of the large

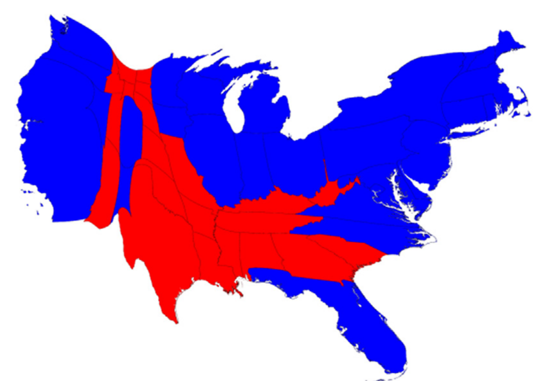

Fig. 2. The 2008 presidential election results weighted by state population. In blue the states won by Obama, in red those by McCain. Retrieved from Marc Newman's University of Michigan's website http://www-personal.umich.edu/~mejn/election/2008/. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

differences that exist across states: blue East and West coast states voting Democrat with red South and Midwestern states voting Republican.[8] Fig. 2 *scales/weights* each state according to its population.[9] Given that East and West Coasts states have a disproportionate mass of citizens relative to Southern or Midwestern states, the blue area now dominates the red one and gives a better representation of the electoral outcome: Obama getting 52.92% of the popular vote to McCain 45.66%, with the remaining votes going to other candidates. Yet Fig. 2 is also not an accurate depiction of the electoral outcome as the President is not directly elected by voters but by the Electoral College. Fig. 3 *scales* each state by its Electoral College vote (ECV) showing Obama's 365 ECVs to McCain's 173. Figs. 2 and 3 are similar – the ECVs of a state are allocated in accordance to the state's proportion of the US population[10] – but not identical

2 In 2008, Obama was the first to use social media in presidential campaigns. He ran a successful fund raising campaign with a portion of his funds having been donated by voters contributing small amounts through social media. Obama also used social media to directly communicate with voters during the campaign.

3 Data on voters' information contains their sociodemographic characteristics (age, education, gender, financial situation, etc.), contact information (home address, emails, phone numbers, social media and twitter accounts, etc.) and answers to past opinion polls on how they would vote where an election held that day.

4 In the past, political campaigns have used emails, pod casts, RRS feeds and cell phone texting to communicate with voters. More recently new technologies such as blogs, YouTube, Twitter, Facebook allow candidates to send messages directly to voters' smart phones and social accounts.

5 Voters' campaign tolerance level measures the number of times they want to ideally be contacted by candidates (see formal definition in Section 2).

6 For example, Texas' sectoral composition includes agriculture, petrochemicals, energy, computers, electronics, aerospace, and biomedical sciences; Massachusetts is a leader in higher education, health care technology, high tech and financial services.

7 This map was taken from Cole's NPR website http://www.npr.org/blogs/itsallpolitics/2012/11/01/163632378/acampaign-map-morphed-by-money.

8 Political differences may originate from historical or cultural differences or from the desire of a state for more independence from the Federal government (Riker, 1964, 1987). Economic differences can arise from endowments of natural resources or from previous regional economic development.

9 This map was taken from Newman's website http://www-personal.umich.edu/~mejn/.

10 Each state is awarded presidential ECVs equal to the number of representatives in the House – which are based on the state's proportion of the US population – plus the number of senators (2 for every state). Maine and Nebraska use the *congressional district method* which selects one elector within each district by popular vote and the remaining two (associated with the two senators) by a statewide popular vote.

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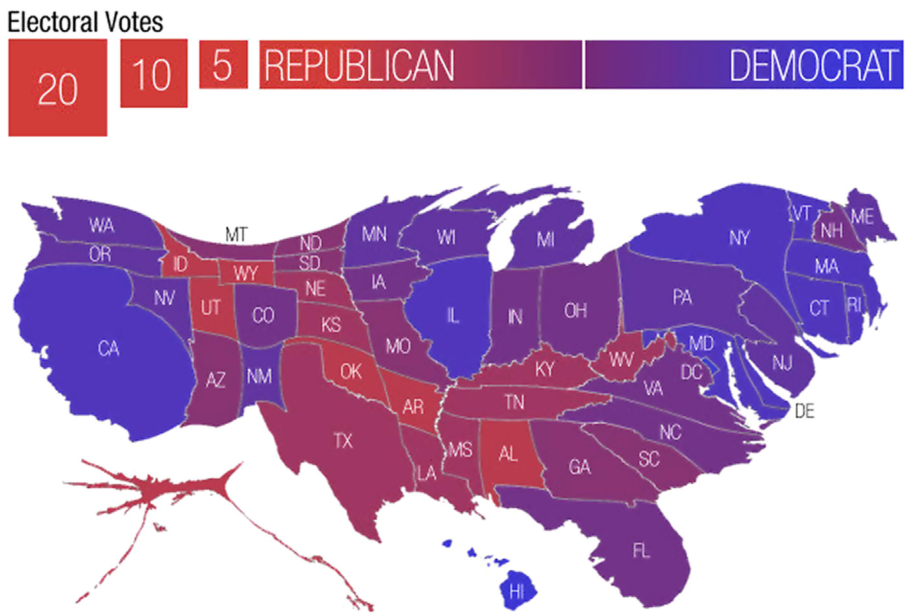

Fig. 3. Cartogram of the 2008 US Presidential election map scaled by Electoral College votes. Obama in blue, McCain in red, in purple states with a 50–50 split in the popular vote. Retrieved from NPR, http://www.npr.org/blogs/itsallpolitics/2012/11/01/163632378/a-campaign-map-morphed-by-money. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

as states with small populations have more ECVs than their population warrants so as not to make the state a dummy player in the election (e.g., Wyoming is twice the size in Fig. 3 than it is in Fig. 2).

These cartograms do not reflect the importance of particular states to candidates in the electoral campaign. Fig. 4 shows the electoral map *scaled* by advertising spending *per state* (in millions of dollars). Even though *all* states are included in the cartogram, only certain states are visible, as candidates spend more time and resources in swing than in safe states or in states they anticipate losing. Fig. 4 illustrates that Obama and McCain did not necessarily campaign in the same states (there are blue and red colored states) and that they both campaigned in swing states (colored in purple). Similarly, Fig. 5 gives the electoral map *scaled* by advertising spending *by state per voter* (in dollars) where, as in Fig. 4, the preponderance of purple swing states is evident. Ad spending per voter was highest in Nevada and New Hampshire.

Figs. 4 and 5 illustrate that candidates' campaigns differ by state, a reflection of candidates' understanding of the economic and political differences that exist across states and their beliefs on their electoral prospects in each state. The figures also illustrate that large sums are spent in swing states where the population is evenly split between the two major candidates with little spent in states where the outcome of the election is undisputed.[11]

Candidates also buy or conduct frequent pre-election polls to gage how voters' perceptions of candidates at the national and state level change through time before and during the electoral campaign. The combination of smart phones, social media and large data sets on detailed voter information now allow candidates to identify the characteristics of committed and undecided voters by state and of swing and uncontested states and to tailor their campaign message to voters and states taking their distinguishing characteristics into account. This implies that there is a symbiotic relationship between state and national campaigns as the national campaign takes differences in voters' preferences across states into account.

The standard spatial model assumes that it is only candidates' *policy positions* that matter to voters. Within the context of the spatial model, controversy has arisen over whether rational candidates will converge to an electoral center, as suggested by Downs (1957) or whether elections are fundamentally unstable, as argued by Riker (1980, 1982, 1986).[12] However, as Stokes (1963, 1992) emphasizes, the non-policy evaluations, or *valences*, of candidates by the electorate are just as important as their policy preferences. Clarke et al. (2009) compare a "Downsian" or spatial model of the 2000 US presidential election with a valence model of the same election, based on voters' perceptions of candidates' traits and find that "the two models have approximately equal explanatory power." As Sanders et al. (2011) comment, valence theory is based on the assumption that "voters maximize their utilities by choosing the party that is best able to deliver policy success." Valence measures the *bias* in favor of one of the party leaders (McKelvey and Patty, 2006). From our work, we argue that neither the Downsian convergence result nor the instability results give an accurate picture of democratic elections. Instead, both position and valence matter in a fundamental way.

When voters' judgments about candidates' competence are modeled as *valences*, the formal model can be linked to Madison's understanding of the nature of the choice of Chief Magistrate (Maddison, 1999 [1787]). Schofield (2002) suggests that Madison's argument on the "extended Republic" may have been influenced by Condorcet's "Jury Theorem" (Condorcet, 1984 [1785]) as it is based on the notion of electoral judgment rather than preference. Recent models involving valence contribute to a Madisonian conception of elections in representative democracies as methods of aggregation of both preferences and judgments.[13]

The empirical voting literature has identified that voters evaluation of candidates depends not only on their policy positions but also on different types of valences. The *traits valence* measures the effect that candidates' traits (popularity, charisma, experience

11 E.g., Obama was expected to win California and Massachusetts and McCain Texas and Oklahoma.

12 See e.g., McKelvey (1976), Schofield (1978), Saari (1997) and Austen-Smith and Banks (1999).

13 See e.g., Ansolabehere and Snyder (2000), Groseclose (2001), Aragones and Palfrey (2002), Schofield (2006) and Zakharov (2009).

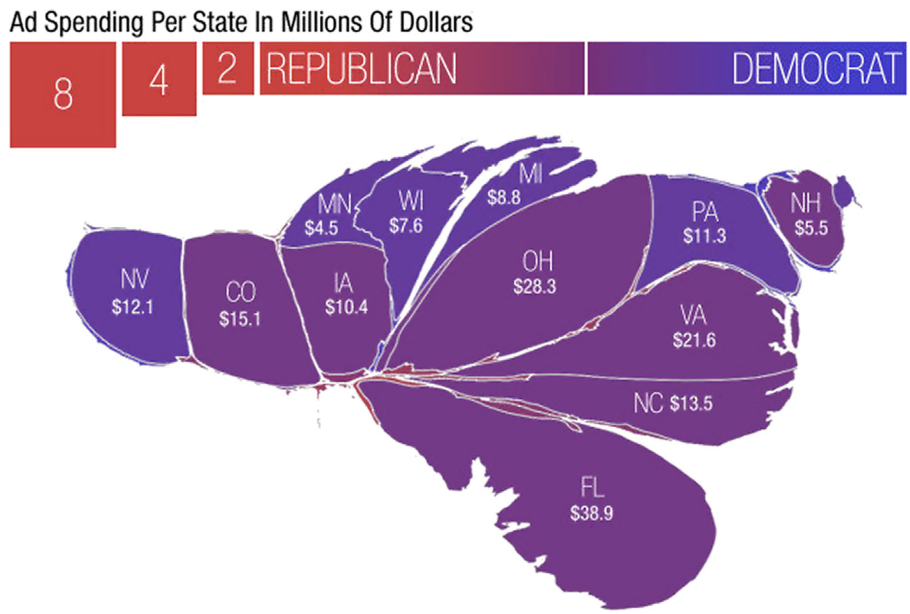

Fig. 4. Cartogram of the 2008 US Presidential election map scaled by ad spending per state (in millions of dollars). Retrieved from NPR, http://www.npr.org/blogs/itsallpolitics/2012/11/01/163632378/a-campaign-map-morphed-by-money. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

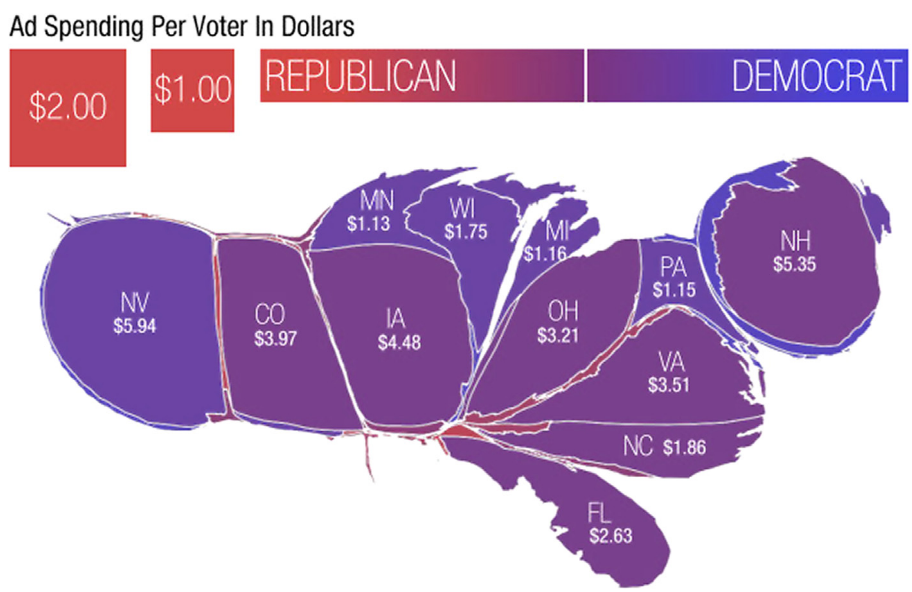

Fig. 5. Cartogram of the 2008 US Presidential election map scaled by ad spending per voter (in dollars). Retrieved from NPR, http://www.npr.org/blogs/itsallpolitics/2012/11/01/163632378/a-campaign-map-morphed-by-money. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

in public office, age, gender, race, etc.) have on voters' choices. The *competency valence* captures voters' beliefs on candidates' ability to govern. The *sociodemographic valence* measures the tendency that voters with certain sociodemographic characteristics have of voting for certain candidates.[14] Evidence indicates that candidates' policy positions and the sociodemographic, traits and competence valences are major determinants of US electoral outcomes (see, e.g., Clarke et al., 2009, Schofield et al., 2011a,b; Gallego and Schofield, 2013, 2016a).

Schofield et al. (2011a,b) study the 2000, 2004 and 2008 US Presidential elections and Kim and Schofield (2016) the 2012 election along the economic and social dimensions without considering the effect that advertising or differences across states have on the election. They find that candidates policy positions were close enough to the electoral mean[15] along the two dimensions as their valence differences were not large enough to induce them to move far away from the electoral mean. In this paper, we examine the effect that campaign advertising has on elections and on candidates' policy positions when differences across states matter.

[14] For example, African–American voters are much more likely to vote for the Democratic candidate than to vote for the Republican one. Thus, Democratic candidates have a higher average sociodemographic valence among African–American voters than do the Republican counterparts.

[15] The electoral mean is the mean of voters' ideal points along each dimension of the policy space.

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In addition, we contend in this paper that models of US presidential elections must also incorporate differences in voting preferences *across states*. The cartograms of the 2008 US Presidential election illustrate large variances in campaign spending across states driven by candidates understanding of inter-state differences in voters' policy and valence preferences. Uncontested states are those where the valence advantage of the state's preferred candidate might be so high that the disadvantaged candidate has no chance of winning, leading candidates to concentrate their campaign efforts in swing states. The cartograms indicate that candidates spend time and resources in contested states paying close attention to the characteristics of voters in these states. We allow the sociodemographic, traits and competence valences to affect voters' choices by incorporating them into voters' utility functions and allow voters' preferences to be state dependent.

Previous election studies in other countries show the effect that regional differences within a country have on elections. Gallego et al. (2014) study the effect that the Bloc Quebecois – running only in Quebec and advocating for greater devolution of powers from the federal government and even independence from Canada – had on the 2004 Canadian election. Labzina and Schofield (2014) examine the effect that the Scottish National Party competing only in Scotland and Plaid Cymru only in Wales had on the 2010 UK general election. These studies highlight that regional differences affect national electoral outcomes because the presence of regional parties influence the electoral outcome at the regional level which in turn affects the electoral outcome at the national level. While there are no regional parties in the US, occasionally independent candidates emerge that change the dynamics of the election (e.g., Nader in the 2000 US election).

We develop a model in which candidates have two campaign instruments – policy and advertising – to show how policy and advertising campaigns are shaped by voters' preferences along these two dimensions. This paper extends Schofield's (2007) national model to one where candidates care about the election at the state and national levels due to differences in voters' preferences across states. In our model, voters are identified by their state of residence, preferred policy, campaign tolerance level, their sociodemographic characteristics, and their perceptions of candidates' traits and competencies. Differences in campaign tolerance levels across voters capture differences in how often they want to be contacted by candidates. Policy and campaign tolerance preferences vary within and between states with the *mean* sociodemographic, trait and competence valences varying only by state. Voters' valences have an idiosyncratic component unobserved by candidates. Since the private component of voters' valences are drawn from Type I Extreme Value Distributions, at the campaign selection stage candidates view voters' utility as random and cannot perfectly anticipate how each individual votes but can estimate their expected vote shares.

In our analysis, we show that national vote shares are a weighted average of the states' vote shares where each state is weight according to its share of the national voting population.[16] Candidates use two electoral instruments – policy and advertising – in an effort to convince voters to vote for them. Their objective is choose the policy and ad campaign that maximize their expected national vote share which depend on the expected vote shares in every state. Thus, in the process of maximizing their expected national vote shares, candidates take differences across states into account. The model examines how voters' preferences affect the policy and ad campaigns choices of candidates and how candidates' campaigns affect their electoral prospects in each state and at the national level.

We show that in the local Nash equilibrium (LNE) of the election, candidates do not locate at the mean of voters' ideal policies or campaign tolerance levels in contrast to Gallego and Schofield (2016b) were differences across states do not matter. Moreover, the policy and advertising campaigns are the weighted averages respectively of voters' ideal policies and campaign tolerance levels in each state multiplied by the weight candidates give to each state along each dimension. Even though the one-person-one-vote principle applies in this model, candidates weight voters in their state and national campaigns and states in their national campaigns differently, giving more weight to undecided voters and states than to committed voters and states. Thus, policies and advertising campaigns are more reflective of the policies and campaign tolerance levels of undecided voters in uncommitted states.

To our knowledge, this is the first model to theoretically explain how candidates' national campaigns depend on voters' characteristics by state and that candidates weight voters and states differently in their policy and advertising campaigns. When voters give different importance to the policy and ad campaign, candidates weight the same voter and the same state differently in their policy and ad campaigns. Voters and states crucial to candidates differ across the policy and advertising dimensions and differ across candidates.

Section 2 gives the preliminaries of the model with the stochastic multi-state election model of the election presented in Section 3. The equilibrium concept used in the analysis is defined in Section 4 and candidates' best response functions leading to the prospective local Nash equilibrium (LNE) of the election in Section 5. Since Section 6 shows that candidates do not converge to a common electoral campaign, Section 7 provides an intuitive explanation of the conditions under which candidates convergence to the LNE derived from the second order conditions given in the Appendix. Our contribution to the literature is discussed in Section 8 with final comments given in Section 9.

2. Preliminaries

We develop a stochastic electoral model in which at least two candidates compete in the election (e.g., Obama, McCain and Nader ran in 2000 election). We model an election in which voters' preferences reflect the political, economic and sociodemographic differences that exist across voters and states and where candidates use two instruments – their policy platform and advertising (or ad) campaign – in their *electoral campaigns*. The ad campaign gives candidates another instrument with which to compete in the election. Since voters perfectly observe candidates' policy announcements, the advertising potion of the campaigns serves *only* to convince voters that they matter to candidates. The ad campaign consists of messages[17] candidates send to voters with campaign advertising costs increasing in the message frequency regardless of the effectiveness of the ad campaign.[18] We study

[16] This is the idea behind the Electoral College vote (ECV) where the state's ECVs are determined by the state's proportion of the US population. We do not incorporate the ECV into our model as we want to avoid the discontinuities introduced by the ECV. Introducing the EVCs requires the use of combinatorial analysis to determine different sets of swing states giving candidates the winning number of ECVs.

[17] Messages are measured in continuous – rather than discrete – terms so that messages of different lengths and frequencies and across states can be compared. If message 1 is twice as long as message 2, then message 1 is assigned a number twice as large as message 2. In this paper, we assume messages are one-dimensional.

[18] An implicit assumption is that candidates have sufficient funds to finance their ad campaigns. In another paper we will examine the effect that activists and their donations have on electoral campaigns.

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candidates' electoral campaign game in response to the anticipated electoral outcome.

Let $\mathcal{C} = \{1, .., j, .., c\}$ be the set of candidates competing in the election in state $k \in \mathcal{S} = \{1, .., k, .., s\}$. Prior to the election, candidates simultaneously announce their electoral campaigns. Candidate j's electoral campaign consists of a policy $z_j \in \mathcal{Z} \subseteq \mathbb{R}$ and an ad message $a_j \in \mathcal{A} \subseteq \mathbb{R}$ for all $j \in \mathcal{C}$. Denote by $\mathcal{K}_j = \mathcal{Z} \times \mathcal{A}$ each candidate's electoral campaign space, and by $\mathcal{K} = \prod_{j\in\mathcal{C}} \mathcal{K}_j$ the campaign space of all candidates.

Let $\mathbf{z}$ and $\mathbf{a}$ be the electoral campaign profile of *all candidates*, i.e.,

$$\mathbf{z} \equiv (z_1, \ldots, z_j, \ldots, z_c) \quad \text{and} \quad \mathbf{a} \equiv (a_1, \ldots, a_j, \ldots, a_c).$$

The vector $\mathbf{z}_{-j}$ (respectively $\mathbf{a}_{-j}$) denotes the profile where all policies and messages *except* j's policy (*message*) are held constant.

Denote by n_k and $\mathcal{N}_k = \{1, .., i, .., n_k\}$ respectively the number and the set of voters in state k and by $n = \sum_{k=1}^{s} n_k$ and $\mathcal{N} = \bigcup_{k=1}^{s} \mathcal{N}_k$ their equivalents at the national level.

Since voters' preferences depend on their state residence, the utility of voter i in state k is given by the vector of utilities i derives from each candidate,

$$\mathbf{U}_i^k(\mathbf{z}, \mathbf{a}) = (u_{i1}^k(z_1, a_1), .., u_{ij}^k(z_j, a_j), \ldots, u_{ic}^k(z_c, a_c)) \quad \text{for all } i \in \mathcal{N}_k \text{ and } k \in \mathcal{S}$$

where the utility voter i in state k derives from candidate j, $u_{ij}^k(z_j, a_j)$, is given by

$$\begin{aligned} u_{ij}^k(z_j, a_j) &= -b_k(x_i - z_j)^2 - e_k(t_i - a_j)^2 + (\mathfrak{s}_j^k + \xi_{ij}^k) \\ &\quad + (\tau_j^k + \varsigma_{ij}^k) + (\lambda_j^k + \epsilon_{ij}^k) \\ &= u_{ij}^{k*}(z_j, a_j) + \xi_{ij}^k + \varsigma_{ij}^k + \epsilon_{ij}^k. \end{aligned} \tag{1}$$

Voter i is characterized by her ideal policy, $x_i \in \mathcal{Z}$, and by the frequency of messages she ideally wants to receive from any candidate, her *campaign tolerance level*, $t_i \in \mathcal{A}$, with (x_i, t_i) drawn from a joint distribution $\mathcal{D}_x^k \times \mathcal{D}_a^k$ for all $i \in \mathcal{N}_k$ and $k \in \mathcal{S}$, so that ideal policies and campaign tolerances vary across voters and states. In (1), u_{ij}^{k*} measures the observable component of u_{ij}^k where the dependence of u_{ij}^k and u_{ij}^{k*} on (x_i, t_i) is taken as understood.

The coefficients b_k and e_k in (1) are positive, the same for all voters in state k, common knowledge and measure the importance voters give to differences with j's policy z_j and ad campaign a_j. Voters have quadratic preferences over policies, i.e., $-b_k(x_i - z_j)^2$ in (1) says that the farther j's policy z_j is from i's ideal policy x_i, the lower is i's utility from j.

Since policies are perfectly observable, the only reason candidates advertise in this model is to give voters a *further* impetus to vote for them and do so by directly contacting voters through ad messages.[19] We assume voters have preferences over how often they want to be contacted by candidates, i.e., are characterized by their *campaign tolerance level*. Voters judge candidates' messages by examining the message frequency compared to their campaign tolerance level and since from the point of view of each voter there can be too much or too little advertising, we model voters as having quadratic preferences over ad campaigns. The *effectiveness of j's ad campaign on voter* i, $-e_k(t_i - a_j)^2$ in (1), depends on the importance voters give to ad campaigns in state k, e_k, and on the messages j sends, a_j, relative to i's campaign tolerance t_i. When $a_j > t_i$ i believes j engaged in too much advertising, an irritant to voter i leading to campaign fatigue[20] and to a lower utility and if $a_j < t_i$, i believes that j's ad campaign reflects j's lack of concern for voters which also lowers i's utility from j.

Voters sociodemographic characteristics affect their choice of candidate. Voter i's *sociodemographic valence* for candidate j in state k is given by $(\mathfrak{s}_j^k + \xi_{ij}^k)$ in (1). The *mean sociodemographic valence* for j in state k, $\mathfrak{s}_j^k$, captures the idea that voters in state k with similar sociodemographic characteristics (gender, age, class, education, financial situation, etc.) share a common evaluation or bias for j. Voters are identified by an h-vector $\mathbf{h}_{ij}^k$ denoting their individual sociodemographic characteristics with mean $\mathbf{h}_j^k \equiv \frac{1}{n_k}\sum_{i\in\mathcal{N}_k} \mathbf{h}_{ij}^k$. The importance sociodemographic characteristics have on the utility i in state k derives from j is modeled by an h-vector γ_j^k *common* to all voters in state k. The composition $\mathfrak{s}_j^k = \{(\gamma_j^k \cdot \mathbf{h}_j^k)\}$ measures the *mean* sociodemographic valence for j in state k. The effect i's sociodemographic characteristics have on i's choice of candidate has an idiosyncratic component, ξ_{ij}^k, not observable by candidates, varying around $\mathfrak{s}_j^k$ according to a Type-I extreme value distribution, $\Upsilon^k(0, \frac{\pi}{6})$, common to all voters in state k with mean zero and variance $\frac{\pi}{6}$.

The effect candidate j's *traits* (race, gender, age, education, charisma, experience in public life, etc.) have on voter i's choice in state k is given by $(\tau_j^k + \varsigma_{ij}^k)$ in (1). The commonly known *mean trait valence* for j in state k, τ_j^k, measures the mean influence j's traits have on voters' choice with the effect on voter i also depending on an idiosyncratic component, ς_{ij}^k. Candidates are identified by a t-vector $\mathbf{t}_j$ capturing their individual traits. The *average* importance voters in state k give to these traits is given by a t-vector $\boldsymbol{\varpi}_j^k$ *common* to all voters in state k. The composition $\tau_j^k = \{(\mathbf{t}_j \cdot \boldsymbol{\varpi}_j^k)\}$ measures the mean effect that j's traits have on voters in state k. We assume that voters' beliefs on the effect candidates' traits have on their utility are *not* perfectly observable by candidates. Voter i's private component, ς_{ij}^k, varies around the mean traits valence, τ_j^k, and is drawn from a Type-I extreme value distribution, $\mathcal{T}^k(0, \frac{\pi}{6})$, common to all voters in state k with zero mean and variance $\frac{\pi}{6}$.

The term $(\lambda_j^k + \epsilon_{ij}^k)$ in (1) represents i's belief on candidate j's competence. The *mean competence valence of voters in state* k, λ_j^k, measures the *common belief* voters in state k have on j's ability to govern. The private portion of i's competence signal, ϵ_{ij}^k, is unobserved by candidates and varies around λ_j^k according to a Type-I extreme value distribution, $\Lambda^k(0, \frac{\pi}{6})$, common across voters in state k with mean zero and variance $\frac{\pi}{6}$.

The three idiosyncratic valence components in voters' utility are drawn independent of each other. Train (2003) shows that the sum of independent Type-I Extreme Value Distributions with identical means and variances also has a Type I Extreme Value Distribution with the same mean and variance. Mathematically, if $\xi_{ij}^k \sim \Upsilon^k(0, \frac{\pi}{6})$, $\varsigma_{ij}^k \sim \mathcal{T}^k(0, \frac{\pi}{6})$ and $\epsilon_{ij}^k \sim \Lambda^k(0, \frac{\pi}{6})$ then their sum $\varkappa_{ij}^k = \xi_{ij}^k + \varsigma_{ij}^k + \epsilon_{ij}^k \sim \mathcal{X}^k(0, \frac{\pi}{6})$ also has a Type-I Extreme value Distribution. In this case, voter i's utility in (1) can be re-written as follows:

$$\begin{aligned} u_{ij}^k(z_j, a_j) &= -b_k(x_i - z_j)^2 - e_k(t_i - a_j)^2 + \sigma_j^k + \tau_j^k + \lambda_j^k + \varkappa_{ij}^k \\ &= u_{ij}^{k*}(z_j, a_j) + \varkappa_{ij}^k. \end{aligned} \tag{2}$$

Since the idiosyncratic valence components of voters' utilities are unobserved by candidates, become known to voters after

[19] We could instead allow candidates to target voters with particular characteristics. Here, we concentrate on the effect that differences across states have on electoral campaigns.

[20] As happens when voters are too frequently contacted by robo-calls or Twitts. Voters vary in the frequency they want to be contacted by candidates, including the number of visits they want from campaign volunteers. Some gladly talk to volunteers or read and re-Twitt the messages they receive, others just close their doors on volunteers or delete the messages they receive.

candidates choose their campaign strategies and since candidates know that these components are drawn from Type-I Extreme Value Distributions, at the electoral campaign selection stage, candidates view voters' utilities as stochastic. Thus, given policies and advertising levels $(\mathbf{z}, \mathbf{a})$ and the idiosyncratic valence distributions Υ^k, $\mathcal{T}^k$, Λ^k and $\mathcal{X}^k$, the probability that voter i in state k chooses candidate j is given by

$$\rho_{ij}^k(z_j, a_j) = \Pr[u_{ij}^k(z_j, a_j) > u_{ih}^k(z_j, a_j), \text{ for all } h \neq j \in \mathcal{C}]$$

where Pr is the probability operator generated by the distribution assumptions on Υ^k, $\mathcal{T}^k$, Λ^k and $\mathcal{X}^k$, and $u_{ij}^k(z_j, a_j)$ is given by (1). The probability that i in state k votes for j is given by the probability that $u_{ij}^k(z_j, a_j) > u_{ih}^k(z_h, a_h)$ for all $h, j \in \mathcal{C}$, i.e., given by the probability that i in state k gets a higher utility from j than from any other candidate.

Since the idiosyncratic valence components and their sum follow Type-I extreme value distributions, the probability that i in state k votes for j has a logit specification, i.e.,

$$\rho_{ij}^k \equiv \rho_{ij}^k(z_j, a_j) \equiv \frac{\exp[u_{ij}^{k*}(z_j, a_j)]}{\sum_{h=1}^{c} \exp[u_{ih}^{k*}(z_j, a_j)]}$$
$$= \left[\sum_{h=1}^{c} \exp[u_{ih}^{k*}(z_h, a_h) - u_{ij}^{k*}(z_j, a_j)]\right]^{-1} \quad (3)$$

for all $j \in \mathcal{C}$, $i \in \mathcal{N}_k$, and $k \in \mathcal{S}$ with the dependence of ρ_{ij}^k on (z_j, a_j) sometimes omitted.

Candidate j's *expected national vote share* is the average of the voting probabilities across all voters in the country, i.e.,

$$V_j(\mathbf{z}, \mathbf{a}) \equiv \frac{1}{n} \sum_{i \in \mathcal{N}} \rho_{ij}^k(z_j, a_j) \quad (4)$$

given ρ_{ij}^k in (3) with expected *national* vote shares adding to 1, $\sum_{j \in \mathcal{C}} V_j(\mathbf{z}, \mathbf{a}) = 1$.

Since voters preferences are state dependent, when we group voters by state, j's expected national vote share in (4) can be re-written as

$$V_j(\mathbf{z}, \mathbf{a}) \equiv \frac{1}{n} \sum_{k \in \mathcal{S}} \sum_{i \in \mathcal{N}_k} \rho_{ij}^k(z_j, a_j)$$
$$= \sum_{k \in \mathcal{S}} \frac{n_k}{n} \frac{1}{n_k} \sum_{i \in \mathcal{N}_k} \rho_{ij}^k(z_j, a_j). \quad (5)$$

Candidate j's *expected vote share in state* k is given by[21]

$$v_j^k(\mathbf{z}, \mathbf{a}) \equiv \frac{1}{n_k} \sum_{i \in \mathcal{N}_k} \rho_{ij}^k(z_j, a_j) \quad (6)$$

where ρ_{ij}^k is given by (3) and where v_j^k is the average probability that voters in state k vote for j with the sum of vote shares in *state* k adding to 1, $\sum_{j \in \mathcal{C}} v_j^k(z_j, a_j) = 1$.

After substituting (6) into (5), j's national vote share becomes

$$V_j(\mathbf{z}, \mathbf{a}) = \sum_{k \in \mathcal{S}} \frac{n_k}{n} v_j^k(\mathbf{z}, \mathbf{a}), \quad (7)$$

i.e., j's expected national vote share is given by the weighted average of j's *state* vote shares where states are weight according to their proportion of the national voting population[22] $\frac{n_k}{n}$. Thus, candidates care about the electoral outcome in every state since voters' preferences are state dependent and state elections determine the outcome at the national level.

Candidates' objective is to maximize their national vote shares and since they view voters' decisions as stochastic, they cannot perfectly anticipate how individuals will vote but can estimate their *expected* vote shares at the state and national levels. Candidate j chooses its policy and ad campaign, (z_j, a_j), to maximize its expected *national* vote share, taking the campaign of other candidates, $(\mathbf{z}_{-j}, \mathbf{a}_{-j})$, as given, that is, j's objective is to

$$\max_{z_j \in \mathcal{Z}, a_j \in \mathcal{A}} V_j(\mathbf{z}, \mathbf{a}) = \frac{1}{n} \sum_{i \in \mathcal{N}} \rho_{ij}^k(z_j, a_j)$$
$$= \sum_{k \in \mathcal{S}} \frac{n_k}{n} v_j^k(\mathbf{z}, \mathbf{a}) \quad \text{for all } j \in \mathcal{C}$$

where the last term follows from (7). This equation highlights that in the process of choosing the policy and advertising levels that maximize its expected national vote share, j takes into account how its choices affect its expected vote share in *every* state.

3. The stochastic multi-state (SMS) electoral model

The timing of events in this stochastic multi-state election model is as follows:

1. Candidates simultaneously announce their policy platforms and advertising campaigns.
2. After observing platforms and advertising levels, each voter learns the idiosyncratic components of their sociodemographic, traits and competence valences.
3. The election takes place.
4. The President elect implements the announced policy platform.

Formally, the election game in this multi-state model is defined as follows.

Definition 1. The stochastic multi-state (SMS) election model can be represented by a game in normal form $\mathcal{G}\langle \mathcal{C}, \mathcal{S}, \mathcal{K}, \mathcal{D}_x^k \times \mathcal{D}_a^k \, \forall k \in \mathcal{S}, \mathbf{V} = (v_j^k \, \forall k \in \mathcal{S} \text{ and } \forall j \in \mathcal{C}, V_j \, \forall j \in \mathcal{C})\rangle$, where

1. **Players**: $\mathcal{C} = \{1, .., j, .., c\}$ is the set of candidates competing in the election.
2. **Electoral Jurisdictions**: $\mathcal{S} = \{1, .., k, .., s\}$ is the set of states in which candidates compete with n_k and $\mathcal{N}_k$ denoting number of voters and the set of voters in state k and n and $\mathcal{N}$ their national counterparts.
3. **Strategies**: $\mathcal{K}_j = \mathcal{Z} \times \mathcal{A}$ denotes the electoral campaign space of candidate j where $\mathcal{Z}$ and $\mathcal{A}$ are the policy and ad messaging spaces, and where the electoral campaign space of all candidates is given by $\mathcal{K} = \prod_{j \in \mathcal{C}} \mathcal{K}_j$.
4. **Voters**: Voters' utilities are state dependent and characterized by their ideal policy and campaign tolerance level, $(x_i, t_i) \in \mathcal{Z} \times \mathcal{A}$ for $i \in \mathcal{N}_k$, by the importance they give to the policy and ad dimensions, (b_k, e_k), and by the mean sociodemographic, traits and competence valences, $(\mathfrak{s}_j^k, \tau_j^k, \lambda_j^k)$ in every state with the distribution of voters' ideal policies and campaign tolerances in state k given by $\mathcal{D}_x^k \times \mathcal{D}_a^k$ for all $k \in \mathcal{S}$. The utility voter i in state k derives from candidate j is given by (1) and subject to three idiosyncratic components drawn from Type I Extreme Value Distributions, $\Upsilon^k(0, \frac{\pi}{6})$, $\mathcal{T}^k(0, \frac{\pi}{6})$ and $\Lambda^k(0, \frac{\pi}{6})$ whose sum $\mathcal{X}^k(0, \frac{\pi}{6})$ also has a Type I Extreme value distribution.

[21] State vote shares are given in lower case with the national ones in upper case.

[22] As is well known, the selection of the US president is determined by the Electoral College Vote (ECV) and indirectly by the state vote shares instead of by the national vote shares. In this paper we do not model the ECV but extend Schofield's (2007) model to one where differences across states matter. By incorporating state differences into the model we are closer to a model that includes the ECV.

5. **Candidates' Payoff Functions**: For any candidate $j \in \mathcal{C}$, j's expected state k and expected national vote share functions are respectively given by $v_j^k : \prod_{j\in\mathcal{C}} \mathcal{K}_j \to [0,1]$ in (6) and by $V_j : \prod_{j\in\mathcal{C}} \mathcal{K}_j \to [0,1]$ in (4) or (7). Let $\mathbf{v}^k(\mathbf{z},\mathbf{a}) = (v_j^k(z_j,a_j)$ for all $j \in \mathcal{C})$ for all $k \in \mathcal{S}$ and $V_j(\mathbf{z},\mathbf{a})$ for all $j \in \mathcal{C}$ respectively represent the payoff function profile of all candidates in state k for all $k \in \mathcal{S}$ and at the national level. So that $\mathbf{V}(\mathbf{z},\mathbf{a}) = (v_j^k(z_j,a_j)$ for all $j \in \mathcal{C}$ and $k \in \mathcal{S}$; $V_j(\mathbf{z},\mathbf{a})$ for all $j \in \mathcal{C})$ represents the vector of candidates' payoff functions.

Let us find the *local Nash equilibria* (LNE) of this Stochastic Multi-State election model.

4. The equilibrium of the stochastic election model

The campaign profile $(\mathbf{z}^*, \mathbf{a}^*)$ is a LNE, if when candidates choose their policies in $\mathbf{z}^*$ and advertising campaigns in $\mathbf{a}^*$, each candidate's expected national vote share function is at a local maximum at $(\mathbf{z}^*, \mathbf{a}^*)$. That is, $(\mathbf{z}^*, \mathbf{a}^*)$ is such that no candidate may either shift its policy *or* ad messaging level by a *small* amount – ceteris paribus – to increase its expected national vote share. Formally,

Definition 2. A strict (weak) local Nash equilibrium of the stochastic multi-state electoral campaign model $\mathcal{G}\langle \mathcal{C}, \mathcal{S}, \mathcal{K}, \mathcal{D}_x^k \times \mathcal{D}_a^k\ \forall k \in \mathcal{S}, \mathbf{V} = (v_j^k\ \forall k \in \mathcal{S}$ and $\forall j \in \mathcal{C}, V_j\ \forall j \in \mathcal{C})\rangle$ is a vector of candidate policies and advertising campaign levels, $(\mathbf{z}^*, \mathbf{a}^*)$ such that for each $j \in \mathcal{C}$:

1. There exists a small neighborhood of z_j^*, $B_j(z_j^*) \subset \mathcal{Z}$, such that for all $z_j' \in B_j(z_j^*) - \{z_j^*\}$

   $V_j(z^*, a^*) > (\geqslant) V_j(z_j', \mathbf{z}_{-j}^*, \mathbf{a}^*)$.
2. There exists a small neighborhood of a_j^*, $B_j(a_j^*) \subset \mathcal{A}$, such that for all $a_j' \in B_j(a_j^*) - \{a_j^*\}$

   $V_j(\mathbf{z}^*, \mathbf{a}^*) > (\geqslant) V_j(\mathbf{z}^*, a_j', \mathbf{a}_{-j}^*)$.

Since the observable component of voter i's utility for candidate j in state k, u_{ij}^{k*} in (1) for all $i \in \mathcal{N}_k$ and $k \in \mathcal{S}$, and the distributions of voters' ideal policies and campaign tolerance levels, $\mathcal{D}_x^k \times \mathcal{D}_a^k$ for all $k \in \mathcal{S}$, are continuously differentiable, then so are ρ_{ij}^k in (3), $v_j^k(\mathbf{z},\mathbf{a})$ in (6) and $V_j(\mathbf{z},\mathbf{a})$ in (4). We can then estimate how candidate j's state and national expected vote shares change if j *marginally* adjusts its policy *or* ad campaign, ceteris paribus.

Remark 1. If in Definition 2 we can substitute $\mathcal{Z}$ for $B_j(z_j^*)$ and $\mathcal{A}$ for $B_j(a_j^*)$ for all $j \in \mathcal{C}$, then a LNE is also a pure strategy Nash equilibrium (PNE) of the election.

Remark 2. In political models, the equilibrium may be characterized by positive eigenvalues for the Hessian of one of the candidates. As a consequence, the expected national vote share functions of such a candidate fail pseudo-concavity. Therefore, none of the usual fixed point arguments can be used to assert existence of a "global" pure Nash equilibrium (PNE). For this reason, we use the concept of a "critical Nash equilibrium" (CNE), namely a vector of strategies which satisfies the first-order condition for a local maximum of candidates' expected national vote share functions. Standard arguments based on the index, together with transversality arguments can be used to show that a CNE will exist and that, generically, it will be isolated.[23] A strict (weak) local Nash equilibrium (LNE) satisfies the first-order condition, together with the second-order condition that the Hessians of all candidates are negative (semi-) definite at the CNE. Clearly, the set of LNE will contain the PNE.

The parameters $(b_k, e_k, \mathfrak{s}_j^k, \tau_j^k, \lambda_j^k)$ for all $j \in \mathcal{C}$ and $k \in \mathcal{S}$ in voter i's utility function in (1) and the distributions $\mathcal{D}_x^k$ and $\mathcal{D}_a^k$ of voters' ideal policies and campaign tolerance levels, for all $k \in \mathcal{S}$, are exogenously given in the model. Using these parameters and the distribution of voters' preferences, any vector of candidate policies and advertising levels, $(\mathbf{z},\mathbf{a})$ can be mapped to a vector of expected vote share functions at the state and national levels, i.e., we can estimate the following vector of expected voters shares

$\mathbf{V}(\mathbf{z},\mathbf{a}) = (v_j^k(\mathbf{z},\mathbf{a})$ for all $j \in \mathcal{C}$ and all $k \in \mathcal{S}$; $V_j(\mathbf{z},\mathbf{a})$ for all $j \in \mathcal{C})$.

Candidate j's expected national vote share function is at a local maximum if the following *first* and *second* order conditions are satisfied. Candidate j's *first order necessary conditions (FONC)* determine j's *best response* policy and advertising functions. With two campaign instruments (policy and advertising) and c candidates, the game generates $2 \times c$ best response functions and their associated $2 \times c$ *critical values*[24] for each candidate.

We then examine whether at these critical values the expected national vote shares functions of each candidate are at a *maximum, minimum or a saddle point*. To do so, we use the *Hessian matrix of second order partial derivatives of the national vote share functions* evaluated at these critical values as the Hessians determine the *local curvature* of the vote share functions at these critical values. The *sufficient* (*necessary*) second order condition (SOC) for j's expected vote shares to be at a maximum at the critical values is that the Hessian be negative (semi-) definite at these critical values which occurs only when the eigenvalues of the Hessian of the vote shares are all negative (non-positive) at these critical values.

Denote the critical (C) policy and ad campaign profile of all candidates by

$$\mathbf{z}^C \equiv (z_1^C, .., z_j^C, .., z_c^C) \quad \text{and} \quad \mathbf{a}^C \equiv (a_1^C, .., a_j^C, .., a_c^C). \tag{8}$$

The profile $(\mathbf{z}^C, \mathbf{a}^C)$ represents a *prospective*[25] LNE of the election.

5. First order necessary conditions (FONC)

We first examine the effect that changes in j's policy or ad messaging frequency have on voters' choices, then study the effect these changes have on j's *expected national vote share*.

5.1. Marginal effects on voters' decisions

The *marginal* impact of a change in j's policy or message, z_j or a_j, on the probability that voter i in state k votes for candidate j is given by *partial* derivative of ρ_{ij}^k in (3) *with respect to* (*wrt*) to each choice variable, *ceteris paribus*, i.e.,

$$\mathbf{D}\rho_{ij}^k(\mathbf{z},\mathbf{a}) \equiv \begin{pmatrix} \dfrac{\partial \rho_{ij}^k}{\partial z_j} \\ \dfrac{\partial \rho_{ij}^k}{\partial a_j} \end{pmatrix} = 2\rho_{ij}^k(1-\rho_{ij}^k) \begin{pmatrix} b_k(x_i - z_j) \\ e_k(t_i - a_j) \end{pmatrix}. \tag{9}$$

The effect of a marginal change in one of candidate j's choice variables on ρ_{ij}^k depends on the *endogenous* probability voter i in

23 Ansolabehere and Snyder (2000) show existence of equilibrium in multidimensional policy spaces with valence.

24 Called *critical* as at these values the expected vote shares functions may not be at a maximum.

25 *Prospective* as it is not yet known if the critical choices satisfy the SOC for a maximum.

$$\begin{pmatrix} z_j^C \\ a_j^C \end{pmatrix} = \sum_{k\in\mathcal{S}}\sum_{i\in\mathcal{N}_k} W_{ij}^{kC}\begin{pmatrix} x_i \\ t_i \end{pmatrix} \quad \text{where} \tag{12}$$

$$W_{ij}^{kC} \equiv \begin{pmatrix} W_{ij}^{kC}(z) \equiv \dfrac{\rho_{ij}^{kC}(1-\rho_{ij}^{kC})b_k}{\sum\limits_{r\in\mathcal{S}}\sum\limits_{i\in\mathcal{N}_k}\rho_{ij}^{rC}(1-\rho_{ij}^{rC})b_r} & 0 \\ 0 & W_{ij}^{kC}(a) \equiv \dfrac{\rho_{ij}^{kC}(1-\rho_{ij}^{kC})e_k}{\sum\limits_{r\in\mathcal{S}}\sum\limits_{i\in\mathcal{N}_k}\rho_{ij}^{rC}(1-\rho_{ij}^{rC})e_r} \end{pmatrix} \tag{13}$$

$$\text{or } W_{ij}^{kC} \equiv \begin{pmatrix} W_{ij}^{kC}(z) \equiv \dfrac{q_{ij}^{kC}b_k}{\sum\limits_{r\in\mathcal{S}}\sum\limits_{i\in\mathcal{N}_k}q_{ij}^{rC}b_r} & 0 \\ 0 & W_{ij}^{kC}(a) \equiv \dfrac{q_{ij}^{kC}e_k}{\sum\limits_{r\in\mathcal{S}}\sum\limits_{i\in\mathcal{N}_k}q_{ij}^{rC}e_r} \end{pmatrix} \tag{14}$$

$$\text{with } q_{ij}^{kC} \equiv \rho_{ij}^{kC}(1-\rho_{ij}^{kC}) \tag{15}$$

and ρ_{ij}^{kC} given by ρ_{ij}^{k} in (3) when evaluated at $(\mathbf{z}^C, \mathbf{a}^C)$, i.e.,

$$\rho_{ij}^{kC} \equiv \rho_{ij}^{k}(\mathbf{z}^C, \mathbf{a}^C). \tag{16}$$

Box I.

state k votes for j, ρ_{ij}^k in (3), and for any other candidate, $(1-\rho_{ij}^k)$; on the importance state k voters give to each dimension, b_k or e_k; and on how far j's corresponding choice variable z_j or a_j is from i's ideal policy or tolerance level, x_i or t_i.

5.2. Candidates' best response functions

Candidate j chooses its policy and ad campaign to maximize its expected *national* vote share $V_j(\mathbf{z}, \mathbf{a})$ in (4) or (7) holding $(\mathbf{z}_{-j}^C, \mathbf{a}_{-j}^C)$ constant. To find j's *national* best response functions take the *partial* derivative of (4) or (7) wrt z_j and a_j and set it equal to zero, i.e.,

$$\begin{aligned}\mathbf{D}V_j(\mathbf{z}, \mathbf{a}) &= \frac{1}{n}\sum_{i\in\mathcal{N}}\mathbf{D}\rho_{ij}^k(\mathbf{z}, \mathbf{a}) = \sum_{k\in\mathcal{S}}\frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\mathbf{D}\rho_{ij}^k(\mathbf{z}, \mathbf{a}) \\ &= \sum_{k\in\mathcal{S}}\frac{n_k}{n}\mathbf{D}v_j^k(\mathbf{z}, \mathbf{a}) = 0. \end{aligned} \tag{10}$$

The last term in (10) – using the state population weighted version of V_j in (7) – shows that j takes into account the effect that j's national choices have on j's vote shares in *every* state.

After substituting (9) into (10), this FONC becomes

$$\mathbf{D}V_j(\mathbf{z}, \mathbf{a}) = \frac{1}{n}\sum_{k\in\mathcal{S}}\sum_{i\in\mathcal{N}_k}2\rho_{ij}^k(1-\rho_{ij}^k)\begin{pmatrix} b_k(x_i - z_j) \\ e_k(t_i - a_j)\end{pmatrix} = 0. \tag{11}$$

Since this FONC is satisfied when

$$\sum_{k\in\mathcal{S}}\sum_{i\in\mathcal{N}_k}\rho_{ij}^k(1-\rho_{ij}^k)\begin{pmatrix} b_k(x_i - z_j) \\ e_k(t_i - a_j)\end{pmatrix} = 0,$$

then after isolating z_j and a_j, j's *national best response functions* are given by[26] equations in Box I.

The *weights* candidate j gives *voter i* in state k in its best response policy and ad messaging functions at $(\mathbf{z}^C, \mathbf{a}^C)$ are given by the diagonal of W_{ij}^{kC} in (13). These weights depend on the importance voters in state k give to each dimension, b_k or e_k, and on the probability that voter i in state k votes for j at (z_j^C, a_j^C), ρ_{ij}^{kC} and for any other candidate, $(1-\rho_{ij}^{kC})$, *relative* to the probability that all other voters in state k and in other states vote for j or for any other candidate, weight by the corresponding importance votes in state k give to each dimension, i.e., depend on $\sum_{r\in\mathcal{S}}\sum_{i\in\mathcal{N}_k}\rho_{ij}^{rC}(1-\rho_{ij}^{rC})b_r$ or on $\sum_{r\in\mathcal{S}}\sum_{i\in\mathcal{N}_k}\rho_{ij}^{rC}(1-\rho_{ij}^{rC})e_r$. Since ρ_{ij}^{kC} depends on the policy and ad messaging profiles of all candidates, j's best response functions (z_j^C, a_j^C) depend on the critical choices of all *other* candidates.

From W_{ij}^{kC} in (13) we can find the weight candidate j gives *state k* in its policy and ad campaign. To see this, multiply W_{ij}^{kC} by the identity matrix expressed as

$$I \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \dfrac{\sum\limits_{i\in\mathcal{N}_k}\rho_{ij}^{rC}(1-\rho_{ij}^{rC})b_r}{\sum\limits_{i\in\mathcal{N}_k}\rho_{ij}^{rC}(1-\rho_{ij}^{rC})b_r} & 0 \\ 0 & \dfrac{\sum\limits_{i\in\mathcal{N}_k}\rho_{ij}^{rC}(1-\rho_{ij}^{rC})e_r}{\sum\limits_{i\in\mathcal{N}_k}\rho_{ij}^{rC}(1-\rho_{ij}^{rC})e_r} \end{pmatrix},$$

to obtain, after re-arranging terms, that W_{ij}^{kC} in (13) can be re-written as given in Box II. Using q_{ij}^{kC} in (15), define w_{ij}^{kC} as

$$w_{ij}^{kC} \equiv \frac{\rho_{ij}^{kC}(1-\rho_{ij}^{kC})}{\sum\limits_{i\in\mathcal{N}_k}\rho_{ij}^{kC}(1-\rho_{ij}^{kC})} = \frac{q_{ij}^{kC}}{\sum\limits_{i\in\mathcal{N}_k}q_{ij}^{kC}}. \tag{17}$$

Multiply w_{ij}^{kC} in (17) separately by $\frac{b_k}{b_k}$ and $\frac{e_k}{e_k}$ then substitute into W_{ij}^{kC} in (13) to obtain equations as in Box III.

[26] All functions and variables with a C superscript are evaluated at candidates' critical campaigns $(\mathbf{z}^C, \mathbf{a}^C)$.

$$W_{ij}^{kC} = \begin{pmatrix} \dfrac{\rho_{ij}^{kC}(1-\rho_{ij}^{kC})b_k}{\sum\limits_{i\in\mathcal{N}_k}\rho_{ij}^{kC}(1-\rho_{ij}^{kC})b_k}\dfrac{\sum\limits_{i\in\mathcal{N}_k}\rho_{ij}^{kC}(1-\rho_{ij}^{kC})b_r}{\sum\limits_{r\in\mathcal{S}}\sum\limits_{i\in\mathcal{N}_k}\rho_{ij}^{rC}(1-\rho_{ij}^{rC})b_r} & 0 \\ 0 & \dfrac{\rho_{ij}^{kC}(1-\rho_{ij}^{kC})e_k}{\sum\limits_{i\in\mathcal{N}_k}\rho_{ij}^{kC}(1-\rho_{ij}^{kC})e_k}\dfrac{\sum\limits_{i\in\mathcal{N}_k}\rho_{ij}^{kC}(1-\rho_{ij}^{kC})e_r}{\sum\limits_{r\in\mathcal{S}}\sum\limits_{i\in\mathcal{N}_k}\rho_{ij}^{rC}(1-\rho_{ij}^{rC})e_r} \end{pmatrix}.$$

Box II.

$$W_{ij}^{kC} \equiv \Omega_j^{kC} w_{ij}^{kC} \quad \text{where} \tag{18}$$

$$\Omega_j^{kC} \equiv \begin{pmatrix} \Omega_j^{kC}(z) \equiv \dfrac{\sum\limits_{i\in\mathcal{N}_k}\rho_{ij}^{kC}(1-\rho_{ij}^{kC})b_k}{\sum\limits_{r\in\mathcal{S}}\sum\limits_{i\in\mathcal{N}_k}\rho_{ij}^{rC}(1-\rho_{ij}^{rC})b_r} & 0 \\ 0 & \Omega_j^{kC}(a) \equiv \dfrac{\sum\limits_{i\in\mathcal{N}_k}\rho_{ij}^{kC}(1-\rho_{ij}^{kC})e_k}{\sum\limits_{r\in\mathcal{S}}\sum\limits_{i\in\mathcal{N}_k}\rho_{ij}^{rC}(1-\rho_{ij}^{rC})e_r} \end{pmatrix} \tag{19}$$

$$\equiv \begin{pmatrix} \Omega_j^{kC}(z) \equiv \dfrac{Q_j^{kC}(z)}{\sum\limits_{r\in\mathcal{S}} Q_j^{rC}(z)} & 0 \\ 0 & \Omega_j^{kC}(a) \equiv \dfrac{Q_j^{kC}(a)}{\sum\limits_{r\in\mathcal{S}} Q_j^{rC}(a)} \end{pmatrix}, \tag{20}$$

$$Q_j^{kC}(z) \equiv \sum_{i\in\mathcal{N}_k}\rho_{ij}^{kC}(1-\rho_{ij}^{kC})b_k \quad \text{and} \quad Q_j^{kC}(a) \equiv \sum_{i\in\mathcal{N}_k}\rho_{ij}^{kC}(1-\rho_{ij}^{kC})e_k, \tag{21}$$

with ρ_{ij}^{kC} given in (16) and q_{ij}^{kC} by (15).

Box III.

Using (12) and (18), j's *critical choices*, (z_j^C, a_j^C), can be expressed as

$$\begin{pmatrix} z_j^C \\ a_j^C \end{pmatrix} = \sum_{k\in\mathcal{S}}\sum_{i\in\mathcal{N}_k} W_{ij}^{kC}\begin{pmatrix} x_i \\ t_i \end{pmatrix}$$
$$= \sum_{k\in\mathcal{S}}\Omega_j^{kC}\sum_{i\in\mathcal{N}_k} w_{ij}^{kC}\begin{pmatrix} x_i \\ t_i \end{pmatrix}. \tag{22}$$

Define the second component of the last term in (22) as the electoral campaign that j would like to run in state k, called j's *desired (D) state k* campaign (z_j^{kD}, a_j^{kD}), i.e., let

$$\begin{pmatrix} z_j^{kD} \\ a_j^{kD} \end{pmatrix} \equiv \sum_{i\in\mathcal{N}_k} w_{ij}^{kC}\begin{pmatrix} x_i \\ t_i \end{pmatrix} \tag{23}$$

where w_{ij}^{kC} is given in (17). This says that even though candidates' run only national campaigns, if j could run separate electoral campaigns in each state, j's desired state k campaign would be given by (z_j^{kD}, a_j^{kD}).[27]

From (23), j's desired state k policy and ad campaign are a weighted average of the ideal policy and campaign tolerance levels of voters in state k where the weight j gives to voter i in state k is the same along the two dimensions[28] and given by w_{ij}^{kC} in (17). The weight w_{ij}^{kC} depends on how likely is i to vote for j in state k, ρ_{ij}^{kC} and for any other candidate, $(1-\rho_{ij}^{kC})$, *relative* to all voters in state k given by $\sum_{i\in\mathcal{N}_k}\rho_{ij}^{kC}(1-\rho_{ij}^{kC})$.

The weight w_{ij}^{kC} in (17) is non-monotonic in the probability that i in state k votes for j, ρ_{ij}^{kC}. To see this, recall from (17) that the numerator of w_{ij}^{kC} is given by $q_{ij}^{kC} \equiv \rho_{ij}^{kC}(1-\rho_{ij}^{kC})$. When i in k votes for j with probability one ($\rho_{ij}^{kC} = 1$ so that i is a *core* supporter) or when i votes for j with probability zero ($\rho_{ij}^{kC} = 0$), then $q_{ij}^{kC} = 0$, implying that j gives these two voters *zero* weight in its policy and ad campaign, i.e., $w_{ij}^{kC} = 0$. When i votes for j with probability $\rho_{ij}^{kC} = \frac{1}{2}$, i is *undecided*, and since $q_{ij}^{kC} = \frac{1}{4}$, j gives i the highest weight in its choice functions. Even though the one-person-one-vote principle applies in this model, candidates give state k voters different weights in their desired state k campaign, e.g., $w_{ij}^{kC}(\rho_{ij}^{kC}=0) = w_{ij}^{kC}(\rho_{ij}^{kC}=1) = 0 < w_{ij}^{kC}(\rho_{ij}^{kC}=\frac{1}{2})$. Thus, j caters to undecided voters in state k by giving them a higher weight in its desired state k campaign.

After substituting j's desired *state k* campaign (z_j^{kD}, a_j^{kD}) from (23) into (22), it is clear that j's critical *national* choices are a weighted average of j's *desired state* campaigns, i.e.,

$$\begin{pmatrix} z_j^C \\ a_j^C \end{pmatrix} = \sum_{k\in\mathcal{S}}\Omega_j^{kC}\begin{pmatrix} z_j^{kD} \\ a_j^{kD} \end{pmatrix}. \tag{24}$$

This results shows that in developing their critical national choices, candidates take the diversity of policy and advertising preferences across states into account.

The other interesting result emanating from (18) is that it shows that the weight candidate j gives *voter* i in j's *national* choices can

[27] Note that (z_j^{kD}, a_j^{kD}) is the campaign that maximizes j's state k vote share when j only cares about the election in state k.

[28] Suppose voters in state k are equally likely to vote for j with probability $\rho_{ij}^k = \upsilon$. The weight j gives each voter in state k is $w_{ij}^k = 1/n_k$, j weights each voter according to the inverse of k's voting population.

be *decomposed* into the product of the weight j gives *voter* i in j's *desired state* k campaign, w_{ij}^{kC} in (17), and the weight j gives *state* k, Ω_j^{kC} in (19), in its *national* choices.

The *weights* candidate j gives to *state* k along the policy and advertising dimensions in its critical choices are given by $\Omega_j^{kC}(z)$ and $\Omega_j^{kC}(a)$ in (19). $\Omega_j^{kC}(z)$ is the weighted probability with which state k votes for j and for any other candidate weight by the importance voters in state k give to the policy dimension, b_k, i.e., $Q_j^{kC}(z)$ in (21), *relative* to all states measured by $\sum_{k\in\mathcal{S}} Q_j^{kC}(z)$, which depends on ρ_{ij}^{kC}. Similarly for $\Omega_j^{kC}(a)$. Since ρ_{ij}^{kC} depends on $(\mathbf{z}^C,\mathbf{a}^C)$, so do $\Omega_j^{kC}(z)$ and $\Omega_j^{kC}(a)$.

The weight state k receives in j's ad campaign, $\Omega_j^{kC}(a)$ in (19), is non-monotonic in the numerator of $\Omega_j^{kC}(a)$, i.e., in $Q_j^{kC}(a)$ in (20). To see this note that when *all* voters in state k vote for j with probability one, so that $\rho_{ij}^{kC}=1$ for all $i\in\mathcal{N}_k$, j expects to win state k with probability 1, i.e., $v_j^{kC}=1$, making k a *perfectly safe* state for j. If *all* voters in k do not vote for j, so that $\rho_{ij}^{kC}=0$ for all $i\in\mathcal{N}_k$, then $v_j^{kC}=0$. In both cases, $Q_j^{kC}(a)=0$ and j gives *zero* weight to these two states in its ad campaign, i.e., $\Omega_j^{kC}(a)=0$. Moreover, when all voters in state k vote for j with probability $\rho_{ij}^{kC}=\frac{1}{2}$, then $v_j^{kC}=\frac{1}{2}$ making k an *undecided* state and $Q_j^{kC}(a)=\frac{1}{4}e_kn_k$. The weight state k receives in j's national ad campaign, $\Omega_j^{kC}(a)$, is then non-monotonic in $Q_j^{kC}(a)$, e.g.,

$$\begin{aligned}\Omega_j^{kC}(a)&\left[Q_j^{kC}(a)(v_j^{kC}=0)=0\right]\\&=\Omega_j^{kC}(a)\left[Q_j^{kC}(a)(v_j^{kC}=1)=0\right]=0\\&<\Omega_j^{kC}(a)\left[Q_j^{kC}(a)\left(v_j^{kC}=\frac{1}{2}\right)=\frac{1}{4}e_kn_k\right].\end{aligned}$$

When state k is undecided, j caters to voters in k by giving state k a higher weight in its ad campaign, a_j^C, than when k is a safe state or if k does not vote for j. Similarly, the weight candidate j gives state k in its national policy, $\Omega_j^{kC}(z)$ in (19), is non-monotonic $Q_j^{kC}(z)\equiv\sum_{i\in\mathcal{N}_k}\rho_{ij}^{kC}(1-\rho_{ij}^{kC})b_k$ – the numerator of $\Omega_j^{kC}(z)$ in (20) – and thus more reflective of the policy preferences of voters in undecided states.

Since j gives a higher weight to undecided voters and states, j's critical national choices are driven by the preferences of uncommitted voters and by the importance voters in undecided states give to each dimension, (b_k,e_k), rather than by the preferences of voters in safe states or of states and voters that care not at all for j.

Candidate j's desired state k ad campaign, a_j^{kD} in (23), measures the number of messages candidate j would like to send voters in state k when j is only concerned with the election in state k. The weights j gives state k in its ad campaign, $\Omega_j^{kC}(a)$ in (19), represent the *relative* frequency with which j advertises in state k. Thus, candidates advertise more frequently in states receiving higher weights in their ad campaign.

We have shown that candidates' critical choices are not set independently of the electoral campaigns they would like to run in every state as critical policies and ad campaigns are a weighted average of those they would like to implement in each state. The weights a state or voter receives in candidates' choices depend on how crucial this state or voter is to the candidate along each dimension. When voters in state k give different importance to policies and ad campaigns, i.e., when $b_k\neq e_k$ in (1), the model predicts that candidates weight the same voter, W_{ij}^{kC} in (13), or state k, Ω_j^{kC} in (19), differently in its policy and ad campaign. Candidate j's crucial voters or states differ across the policy and advertising dimensions.

The weights j gives voters in its desired state campaign, w_{ij}^{kC} in (17) and in its critical choices, W_{ij}^{kC} in (13), and states in its critical choices, Ω_j^{kC} in (19), are endogenously determined since they pin down candidates' campaigns, $(\mathbf{z}^C,\mathbf{a}^C)$, *and* depend on the probability that voters in state k choose j, ρ_{ij}^{kC} in (16) for all $i\in\mathcal{N}_k$ and $k\in\mathcal{S}$, which depend on candidates' electoral campaigns, $(\mathbf{z}^C,\mathbf{a}^C)$. The weights vary across elections as the parameters in voters' utility, $(b_k,e_k,\mathfrak{s}_j^k,\tau_j^k,\lambda_j^k)$ for all $j\in\mathcal{C}$ and $k\in\mathcal{S}$ in (1) and the distribution of voters' ideal policies and campaign tolerances, $\mathcal{D}_x^k\times\mathcal{D}_a^k$ for all $k\in\mathcal{S}$, vary across elections.

6. Do candidates converge to a common campaign?

The objective is to find candidates' local Nash equilibrium (LNE) policies and advertising campaigns $(\mathbf{z}^*,\mathbf{a}^*)$. We now show that candidates adopt *different* electoral campaigns.

Lemma 1. *There is no convergence to a common electoral campaign.*

Proof. *By contradiction*. Assume candidates adopt identical campaigns. The gist of the proof is as follows. Voter i's utility difference between two candidates is independent of candidates' campaigns and of voters' ideal policy and campaign tolerances, implying that voters in state k vote with equal probability for each candidate. However, candidates weight voters differently in their electoral campaigns due to differences across states emanating from variations in the distribution of voters' policy and advertising preferences, in valences differences and in the importance voters give to each dimension. Candidates adopt then different campaigns, a contradiction of our initial assumption.

Suppose candidates adopt identical campaigns, i.e., $(z_j,a_j)=(z^O,a^O)$ for all $j\in\mathcal{C}$. The probability that i chooses j in state k, ρ_{ij}^{kO} in (3) at (z^O,a^O) depends on the *difference* between the observable component of i's utility from candidates h and j, i.e., using (1)

$$\begin{aligned}u_{ih}^{k*O}-u_{ij}^{k*O}&\equiv u_{ih}^{k*}(z^O,a^O)-u_{ij}^{k*}(z^O,a^O)\\&=(\mathfrak{s}_h^k-\mathfrak{s}_j^k)+(\tau_h^k-\tau_j^k)+(\lambda_h^k-\lambda_j^k)\end{aligned}\tag{25}$$

since the policy and ad campaign components cancel out.

Using (25), the probability that voter i in state k chooses candidate j is independent of candidates' policy and ad campaigns as ρ_{ij}^k in (3) when evaluated at $(\mathbf{z}^O,\mathbf{a}^O)$ reduces to

$$\begin{aligned}\rho_j^{kO}&=\left[\sum_{h\neq j,h=1}^{c}\exp[u_{ih}^{kO*}-u_{ij}^{kO*}]\right]^{-1}\\&=\left[\sum_{h\neq j,h=1}^{c}\exp[(\mathfrak{s}_h^k-\mathfrak{s}_j^k)+(\tau_h^k-\tau_j^k)+(\lambda_h^k-\lambda_j^k)]\right]^{-1},\end{aligned}\tag{26}$$

i.e., ρ_j^{kO} depends only on the difference between the mean sociodemographic, traits and competence valences between candidates in state k and not on their campaign strategies. Clearly, (26) implies that voters in state k choose j with *equal* probability as ρ_j^{kO} does not depend on their ideal policies or campaign tolerance levels, $\rho_{ij}^{kO}=\rho_j^{kO}$ for all $i\in\mathcal{N}_k$. Thus, voters in state k are weighted equally in j's electoral campaign since w_{ij}^{kC} in (17) reduces to

$$w^{kO}\equiv\frac{\rho_j^{kO}(1-\rho_j^{kO})}{\sum_{i\in\mathcal{N}_k}\rho_j^{kO}(1-\rho_j^{kO})}=\frac{1}{n_k},\tag{27}$$

i.e., j weights each voter in state k according to the inverse of state k's voting population.

The weights j gives voter i in its critical choices, W_{ij}^{kC} in (13) at $(\mathbf{z}^O, \mathbf{a}^O)$ are given by

$$W_j^{kO} \equiv W_{ij}^{kC}(\mathbf{z}^O, \mathbf{a}^O) \equiv \frac{1}{n_k}\begin{pmatrix} \dfrac{\rho_j^{kO}(1-\rho_j^{kO})b_k}{\sum_{r\in\mathcal{S}}\rho_j^{rO}(1-\rho_j^{rO})b_r} & 0 \\ 0 & \dfrac{\rho_j^{kO}(1-\rho_j^{kO})e_k}{\sum_{r\in\mathcal{S}}\rho_j^{rO}(1-\rho_j^{rO})e_r}\end{pmatrix} = w^{kO}\Omega_j^{kO} \tag{28}$$

where Ω_j^{kO} is given by Ω_j^{kC} when evaluated at $(\mathbf{z}^O, \mathbf{a}^O)$. The weight Ω_j^{kO} varies across states since the importance voters give to each dimension, b_k and e_k, the weights candidates give to voters in each state, $w^{kO} = \frac{1}{n_k}$, and the probability with which voters vote for the candidates, ρ_j^{kO}, also vary by state. Therefore, the weight j gives voters in its national choices, W_j^{kO}, also varies by state and since ρ_j^{kO} varies across candidates so does W_j^{kO}.

After substituting W_j^{kO} in (28) into (12), j's national best response functions become

$$\begin{pmatrix} z_j^C \\ a_j^C \end{pmatrix} = \sum_{k\in\mathcal{S}} W_j^{kO} \sum_{i\in\mathcal{N}_k}\begin{pmatrix} x_i \\ t_i\end{pmatrix}. \tag{29}$$

With W_j^{kO} varying across candidates and states and since the distribution of voters' preferences also vary across states, candidates do *not* converge to the same campaign, contrary to what was assumed at the beginning of this proof. ■

Having determined that candidates do not adopt the same campaign, we now examine if candidates adopt the *critical* campaigns, $(\mathbf{z}^C, \mathbf{a}^C)$.

7. Convergence to the critical campaigns

We now derive the conditions under which j *adopts* its critical campaign, (z_j^C, a_j^C) assuming that all other candidates adopt their critical campaigns, i.e., holding $(\mathbf{z}_{-j}^C, \mathbf{a}_{-j}^C)$ constant.

The Appendix shows that the LNE of the election is characterized by two pivotal probabilities for candidate j for each voter from which two pivotal state k vote shares are derived for candidate j identifying the *necessary* and *sufficient* conditions under which j adopts (z_j^C, a_j^C) as its campaign in state k. Then, using these two pivotal state k vote shares we derive j's two national pivotal vote shares giving the *necessary* and *sufficient* conditions under which j adopts (z_j^C, a_j^C) as its electoral campaign in all states, i.e., as its electoral campaign at the national level. Before summarizing the results presented in the Appendix, we give the following definitions that facilitate the discussion of these results.

Definition 3. Denote voter *i's necessary and sufficient pivotal probabilities for candidate j in state k* at (z_j^C, a_j^C) by

$$\psi_{ij}^{kC} \equiv \psi_{ij}^{k}(\mathbf{z}^C, \mathbf{a}^C) \equiv \frac{1}{2} \tag{30}$$

$$\varphi_{ij}^{kC} \equiv \varphi_{ij}^{k}(\mathbf{z}^C, \mathbf{a}^C) \equiv \frac{1}{2} - \frac{1}{4}\frac{Tr(\boldsymbol{\iota}_k)}{Tr\left(\boldsymbol{\delta}_{ij}^{kC}\right)} \tag{31}$$

where the importance voters give to each dimension in state k are represented by $\boldsymbol{\iota}_k$ in

$$\boldsymbol{\iota}_k \equiv \begin{pmatrix} b_k & 0 \\ 0 & e_k\end{pmatrix} \quad\Rightarrow\quad Tr(\boldsymbol{\iota}_k) = b_k + e_k. \tag{32}$$

The trace of $\boldsymbol{\iota}_k$, $Tr(\boldsymbol{\iota}_k)$, also given in the Appendix in (54), measures the aggregate importance voters in state k give to the policy and ad dimensions *relative* to the valence components.

The matrix $\boldsymbol{\delta}_{ij}^{kC}$ in (31), given by

$$\boldsymbol{\delta}_{ij}^{kC} \equiv \begin{pmatrix} b_k(x_i - z_j^C)^2 b_k & b_k(x_i - z_j^C)(t_i - a_j^C)e_k \\ b_k(x_i - z_j^C)(t_i - a_j^C)e_k & e_k(t_i - a_j^C)^2 e_k\end{pmatrix}, \tag{33}$$

measures the weighted variance–covariance matrix of voter i's preferences in state k around j's critical choices (z_j^C, a_j^C) where its trace measuring the aggregate variance of i's preferences around j's critical choices is always positive and given by

$$Tr\left(\boldsymbol{\delta}_{ij}^{kC}\right) = b_k(x_i - z_j^C)^2 b_k + e_k(t_i - a_j^C)^2 e_k. \tag{34}$$

Candidate j's expected state k vote share in (6) evaluated at $(\mathbf{z}^C, \mathbf{a}^C)$ is given by

$$v_j^{kC} \equiv \frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\rho_{ij}^{kC}. \tag{35}$$

Define j's sufficient and necessary state k pivotal vote shares at $(\mathbf{z}^C, \mathbf{a}^C)$, Ψ_j^{kC} and Φ_j^{kC}, as[29]

$$\Psi_j^{kC} \equiv \frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\psi_{ij}^{kC} \equiv \frac{1}{2} \tag{36}$$

$$\Phi_j^{kC} \equiv \frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\varphi_{ij}^{kC} \equiv \frac{1}{2} - \frac{1}{4}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\frac{Tr(\boldsymbol{\iota}_k)}{Tr\left(\boldsymbol{\delta}_{ij}^{kC}\right)} \tag{37}$$

where $Tr\left(\boldsymbol{\delta}_{ij}^{kC}\right)$ is given by (34) and $Tr(\boldsymbol{\iota}_k)$ by (32).

Using v_j^{kC} in (35), denote j's expected national vote share in (7) at $(\mathbf{z}^C, \mathbf{a}^C)$ by

$$V_j^C \equiv \sum_{k\in\mathcal{S}}\frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\rho_{ij}^{kC} = \sum_{k\in\mathcal{S}}\frac{n_k}{n}v_j^{kC}. \tag{38}$$

Define j's sufficient and necessary national pivotal vote shares[30] at $(\mathbf{z}^C, \mathbf{a}^C)$, as

$$\eta_j^C \equiv \sum_{k\in\mathcal{S}}\frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\Psi_j^{kC} \equiv \frac{1}{2} \tag{39}$$

$$\vartheta_j^C \equiv \sum_{k\in\mathcal{S}}\frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\Phi_j^{kC} \equiv \frac{1}{2} - \frac{1}{4}\sum_{k\in\mathcal{S}}\frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\frac{Tr(\boldsymbol{\iota}_k)}{Tr\left(\boldsymbol{\delta}_{ij}^{kC}\right)} \tag{40}$$

where $Tr\left(\boldsymbol{\delta}_{ij}^{kC}\right)$ by (34) and $Tr(\boldsymbol{\iota}_k)$ by (32).

Lemma 2 (*Comparing the Sufficient and Necessary Conditions*).

(i) *Voter i's sufficient pivotal probability is always higher than i's necessary pivotal probability, i.e.,* $\psi_{ij}^{kC} > \varphi_{ij}^{kC}$ *for all* $i \in \mathcal{N}_k$, $j \in \mathcal{C}$ *and* $k \in \mathcal{S}$.

(ii) *Candidate j's sufficient state k pivotal vote share is always higher than j's necessary state k pivotal vote share, i.e.,* $\Psi_j^{kC} > \Phi_j^{kC}$ *for all* $j \in \mathcal{C}$ *and* $k \in \mathcal{S}$.

(iii) *Candidate j's sufficient national pivotal vote share is always higher than j's necessary national pivotal vote share, i.e.,* $\eta_j^C > \vartheta_j^C$ *for all* $j \in \mathcal{C}$.

29 Since Φ_j^{kC} depends on j's critical campaign, (z_j^C, a_j^C), it varies across candidates.

30 Since ϑ_j^C depends on j's critical campaign, (z_j^C, a_j^C), it varies across candidates.

Proof. Obvious.

Lemma 2 shows that voter i's sufficient pivotal probability, ψ_{ij}^{kC} in (30), is harder to meet than the necessary one, φ_{ij}^{kC} in (31), that the sufficient condition for candidates to adopt their critical campaign in state k, Ψ_j^{kC} in (36), is more stringent than the necessary one, Φ_j^{kC} in (37), and that the sufficient condition for candidates to adopt their critical campaign at the national level, η_j^C in (39), is more stringent than the necessary one, ϑ_j^C in (40).

The following proposition gives the sufficient and necessary conditions for the critical campaign vector, $(\mathbf{z}^C, \mathbf{a}^C)$, to be a local Nash equilibrium (LNE) of the electoral game.

Proposition 1 (*Convergence to the Critical Campaigns*). *The local Nash equilibrium of the electoral campaign game, $\mathcal{G}\langle \mathcal{C}, \mathcal{S}, \mathcal{K}, \mathcal{D}_x^k \times \mathcal{D}_a^k\, \forall k \in \mathcal{S}, \mathbf{V} = (v_j^k\, \forall k \in \mathcal{S}$ and $\forall j \in \mathcal{C}, V_j\, \forall j \in \mathcal{C})\rangle$ is characterized by the following sufficient and necessary conditions.*

- *If $\eta_j^C \leqslant V_j^C$ for all $j \in \mathcal{C}$, then the **sufficient** condition for convergence to candidates' critical campaign, $(\mathbf{z}^C, \mathbf{a}^C)$, has been met by all candidates. Candidates' critical choices, $(\mathbf{z}^C, \mathbf{a}^C)$, are then a **strict** LNE of the election.*
- *If $\vartheta_j^C \leqslant V_j^C < \eta_j^C$ for some $j \in \mathcal{C}$ and $\eta_h^C \leqslant V_h^C$ for $h \neq j \in \mathcal{C}$, then the **necessary** but not the sufficient condition for convergence to $(\mathbf{z}^C, \mathbf{a}^C)$ has been met by j. Candidates' critical campaigns, $(\mathbf{z}^C, \mathbf{a}^C)$, constitute a **weak** LNE of the election.*
- *If $V_j^C < \vartheta_j^C$ for some $j \in \mathcal{C}$, the necessary condition for convergence to (z_j^C, a_j^C) has **not** been met by j. Candidate j has an incentive to move away from (z_j^C, a_j^C) to increase its vote share as may other candidates. Candidates' critical campaigns, $(\mathbf{z}^C, \mathbf{a}^C)$, are **not** a LNE of the election.*

Proof. See the Appendix.

The intuition behind Proposition 1 follows the proofs provided in the Appendix assuming that all other candidates adopt their critical campaigns, i.e., given $(\mathbf{z}_{-j}^C, \mathbf{a}_{-j}^C)$. We first give the intuition for the sufficient condition then that for the necessary condition.

The *sufficient* condition for j to adopt its critical campaign, (z_j^C, a_j^C), as its campaign – also given in the Appendix in (104) – is that

$$\eta_j^C = \frac{1}{2} < V_j^C \tag{41}$$

where V_j^C is given by (38) and η_j^C by (39). Condition (41) says that j's national vote share at $(\mathbf{z}^C, \mathbf{a}^C)$, V_j^C, must be *high* enough for j to adopt (z_j^C, a_j^C) as its campaign, i.e., V_j^C must be higher than j's *sufficient* national pivotal vote share η_j^C. When (41) is satisfied for all $j \in \mathcal{C}$, the critical campaigns $(\mathbf{z}^C, \mathbf{a}^C)$ are a *strict* LNE of the election.

Condition (41) will be satisfied when there are *enough* states for whom the *sufficient* condition in state k given by

$$\Psi_j^{kC} = \frac{1}{2} < v_j^{kC} \tag{42}$$

is also satisfied where v_j^{kC} is given by (35) and Ψ_j^{kC} by (36). This condition says that the probability of choosing j at $(\mathbf{z}^C, \mathbf{a}^C)$ must be *high enough in state* k, meaning that j's expected state k vote share at $(\mathbf{z}^C, \mathbf{a}^C)$, v_j^{kC} in (35), must be *higher* than j's sufficient *state k* pivotal vote share, Ψ_{ij}^{kC} in (36) for j to adopt (z_j^C, a_j^C) as it campaign in state k.

Moreover, for condition (42) to be fulfilled in *state* k the probability of voting for j must be *high enough for enough voters in state* k, meaning that there must be enough voters for whom the probability that i in state k votes for j, ρ_{ij}^{kC} in (16), is greater than i's sufficient pivotal probability, φ_{ij}^{kC} in (31). That is, it must be the case that for enough voters in state k, the *sufficient condition on voter i in state k for candidate j* is met, i.e., that

$$\psi_{ij}^{kC} = \frac{1}{2} < \rho_{ij}^{kC}. \tag{43}$$

Averaging the left hand side (LHS) of (43) over voters in state k gives j's *sufficient state k pivotal vote share*, Ψ_{ij}^{kC} in (36) then doing a population weighted average of Ψ_{ij}^{kC} over states, i.e., weighting states by $\frac{n_k}{n}$, gives j's *necessary national pivotal vote share*, η_j^C in (39). Averaging the right hand side (RHS) of (43) over voters in state k gives j's expected state k vote share, v_j^{kC} in (35), then doing a population weighted average of v_j^{kC} over states gives j's expected national vote share, V_j^C in (38).

When $\psi_{ij}^{kC} < \rho_{ij}^{kC}$ in (43) is satisfied for enough voters in state k so that $\psi_{ij}^{kC} < v_j^{kC}$ in (42) is satisfied in state k and if $\Psi_{ij}^{kC} < v_j^{kC}$ is satisfied in enough states, then j's national vote share is greater than j's national pivotal vote share, i.e., $\eta_j^C < V_j^C$ in (44) is satisfied for candidate j and j adopts (z_j^C, a_j^C) as its campaign. Additionally, if these three conditions are satisfied for all $j \in \mathcal{C}$, candidates' critical campaigns, $(\mathbf{z}^C, \mathbf{a}^C)$, correspond to a *strict* LNE of the election. Note that these three conditions do not have to be satisfied for the same voters or across the same states for all candidates.

The *necessary* national convergence condition for candidate j to adopt (z_j^C, a_j^C) as its campaign – also given in the Appendix in (67) – is that

$$\vartheta_j^C < V_j^C \tag{44}$$

where V_j^C is given by (38) and ϑ_j^C by (37). In words, this says that the necessary national condition for j to adopt (z_j^C, a_j^C) as its campaign is that j's expected vote share at $(\mathbf{z}^C, \mathbf{a}^C)$, V_j^C, be *high* enough, i.e., is greater than j's necessary national pivotal vote share, ϑ_j^C.

For condition (44) to be satisfied it must be that there are *enough* states for whom the *necessary* condition in state k given by

$$\Phi_j^{kC} < v_j^{kC} \tag{45}$$

is satisfied, i.e., that the probability of choosing $j$at $(\mathbf{z}^C, \mathbf{a}^C)$ must be *high enough* in state k. In words, j adopts (z_j^C, a_j^C) as its campaign in state k when j's expected state k vote share at $(\mathbf{z}^C, \mathbf{a}^C)$, v_j^{kC} in (35), is *higher* than j's necessary *state k* pivotal vote share, Φ_{ij}^{kC} in (37).

Additionally, condition (45) will be satisfied in *state* k when the probability of voting for j is *high enough for enough voters* in state k, meaning that for these voters the probability that i votes for j in state k, ρ_{ij}^{kC} in (16), must be greater than i's necessary pivotal probability, φ_{ij}^{kC} in (31). Therefore, the necessary condition for j to adopt (z_j^C, a_j^C) as it campaign in state k is that there are enough voters in state k for whom the *necessary condition on voter i for candidate* j is met, i.e., that

$$\varphi_{ij}^{kC} < \rho_{ij}^{kC}. \tag{46}$$

Averaging the LHS of (46) over voters in state k gives j's *necessary state k pivotal vote share*, Φ_{ij}^{kC} in (37) then doing a population weighted average of Φ_{ij}^{kC} over states gives j's *necessary national pivotal vote share*, ϑ_j^C in (40). When averaging the RHS of (46) over voters in state k we obtain j's expected state k vote share, v_j^{kC} in (35), then doing a population weighted average of v_j^{kC} over states gives j's expected national vote share, V_j^C in (38).

Thus, when $\varphi_{ij}^{kC} < \rho_{ij}^{kC}$ in (46) is satisfied for enough voters in state k so that $\Phi_{ij}^{kC} < v_j^{kC}$ in (45) is satisfied in state k and if $\Phi_{ij}^{kC} < \upsilon_j^{kC}$ is satisfied in enough states, then *j*'s national vote share is greater than *j*'s necessary national pivotal vote share, i.e., $\vartheta_j^C < V_j^C$ in (44) and *j* adopts (z_j^C, a_j^C) as it campaign. When $\vartheta_j^C < V_j^C$ in (44) is satisfied for all $j \in \mathcal{C}$, these three conditions are also satisfied for all $j \in \mathcal{C}$, candidates' critical campaigns, $(\mathbf{z}^C, \mathbf{a}^C)$, are a *weak* LNE of the election. These three conditions do not have to be satisfied for the same voters and in the same states for all candidates.

When $V_j^C < \vartheta_j^C$, candidate *j*'s expected national vote share does *not* meet the necessary convergence condition in (44), *j* moves away from (z_j^C, a_j^C) to increase its votes since *j*'s national vote share is at a minimum or at a saddle-point at $(\mathbf{z}^C, \mathbf{a}^C)$. If *j* moves away from either its policy or ad campaign to increase its votes, other candidates may also find it in their interest to change their electoral campaigns. Candidates' critical campaigns, $(\mathbf{z}^C, \mathbf{a}^C)$, are then *not* a LNE of the election.

Predictions emanating from these results are that if candidates (and their campaign strategists) realize that candidates' policies or ad campaigns are ineffective, in the sense of convincing voters to vote for them, then adjustments to policies and/or ad campaigns are made in an effort to increase the probability that voters vote for candidate *j*.

8. The literature

Recent literature focuses on the effects of campaign expenditure on US elections ignoring differences across states. Meirowitz (2008) models candidates as selecting their campaign effort (time, energy and money) to win elections to show that marginal asymmetries in costs or technology can explain incumbency advantage ignoring the policy side of the election. Herrerra et al. (2008) find that greater volatility in voters' policy preferences forces the two parties to spend more on the election. Ashworth and Bueno de Mesquita (2009) assume that candidates buy valence to increase their electoral chances. In our model, asymmetries between candidates are generated by the effectiveness of the policy and ad campaigns on each voter at the state level and by differences in valences across states. Moreover, the distribution of voters' policy and advertising preferences within a state affect their pivotal probabilities which together with the distribution of voters' preferences across states affect candidates' pivotal vote shares at the state and national levels and this affects the LNE of the election and the electoral outcome. In our model, candidates' messages exert a differential impact on voters' choices within and across states and thus affects candidates' policy and ad campaigns.

Models studying the effect of *ad campaigns* on US Presidential elections at the *state level* ignore the policy side of the campaign. Brams and Davis (1973) find that under direct popular vote, campaign resources are allocated in proportion to the number of uncommitted voters in each state. In Snyder's (1989) two-party legislative election model parties allocate campaign resources across districts to maximize either the expected number of seats or the probability of winning a majority of seats. Strömberg (2008) develops a model of the allocation of campaign resources (daily visits) across states when candidates' maximize the probability of winning the election under direct vote when voters' preferences vary across states and are subject to common state and national shocks after candidates' choose their campaigns. In our model, candidates' electoral campaign consists of policies and ad messages with voters characterized by their ideal policy and campaign tolerance preferences and by individual shocks that vary around the mean sociodemographic, traits and competence valences.

9. Conclusion

This paper generalizes Schofield (2007) in several directions. We allow candidates to use two electoral campaign instruments – policy and advertising – in their bid for office in a world where voters' preferences depend on their state of residence, on policies, advertising campaigns and on the mean sociodemographic, traits and competency valences that are subject to three idiosyncratic shocks. This model is an extension of Gallego and Schofield's (2016b) policy and advertising campaign model to one where differences across states matter.

In the local Nash equilibrium, candidate *j*'s policy and ad campaign depends on the campaigns adopted by all other candidates and on voters' ideal policies and campaign tolerance levels. The equilibrium also shows that since voters' preferences are state dependent, candidates' campaigns take differences in voters' preferences across states into account. Policy and ad campaigns are a weighted average of voters' ideal policies and campaign tolerance levels where the weight of each voter depends on the probability that the voter votes for the candidate relative to the probability that all voters vote for the candidate.

Our results show that the weight candidate *j* gives voter *i* in state *k* in their campaign can be decomposed into the weight *j* would give *i* in *j*'s desired state *k* campaign and the weight *j* gives state *k* in their national campaign. Moreover, our analysis also shows that even though the one-person-one-vote principle applies in this model, candidates weight voters in their desired state and national campaigns and states in their national campaigns differently, giving more weight to undecided voters and states than they do to committed voters and states. To our knowledge, this is the first paper showing that campaigns weight voters and states differently and that the weight given to each voter and state vary across candidates.

The introduction of campaign messages and of voters' campaign tolerance levels is novel and allows us to show that candidates must not only find an optimal policy with which to run in the election but also that they must find the optimal message frequency in order not to alienate to many voters. It shows that candidates may not necessarily adjust their policy platforms but that they can instead adjust their ad messages to increase their vote shares.

We derive the necessary and sufficient conditions for candidates to adopt their critical campaigns and characterize these conditions in very intuitive terms showing that the weak and strict local Nash equilibrium conditions are characterized each by a pivotal vote share for every voter for each candidate leading to a pivotal vote share in each state from which a pivotal national vote shares are derived for each candidate. Proposition 1 summarizes the necessary and sufficient conditions for candidates to adopt their critical campaigns expressed in terms of pivotal vote shares. This proposition shows that for *j* to adopt (z_j^C, a_j^C) as its critical campaign in both the necessary and sufficient conditions there must be enough states voting for candidate *j* with high enough probability which happens only when there are enough voters in these states who vote with high enough probability for *j*. When the sufficient (necessary) condition is satisfied for all (at least one) candidates, $(\mathbf{z}^C, \mathbf{a}^C)$ is a strict (weak) LNE of the election.

The story developed in this model together with the pivotal vote shares and the differential weights candidates give voters and states provide a plausible rationale for the campaign spending patterns observed in the cartograms of the 2008 US Presidential campaign presented in Figs. 1–5.

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Appendix. Second order conditions (SOC)

Section 5 derived candidates' best response campaign functions. We now determine the conditions under which candidates' national vote share functions, V_j in (4) or in (5) for all $j \in \mathcal{C}$, are at a local maximum at $(\mathbf{z}^C, \mathbf{a}^C)$, i.e., for $(\mathbf{z}^C, \mathbf{a}^C)$ to be a LNE of the election.

A.1. Proof of Proposition 1

From Section 5 we know that candidate j's critical choices, (z_j^C, a_j^C), are given in (12). We use the *Hessian matrix of second order partial derivatives* of j's national vote share function to determine if V_j in (4) or in (5) is at a maximum at (z_j^C, a_j^C), holding $(\mathbf{z}_{-j}^C, \mathbf{a}_{-j}^C)$ constant. To find this Hessian we need the second order partial derivatives of the probability that voter i in state k votes for candidate j, $\mathbf{D}^2\rho_{ij}^k$, when evaluated at $(\mathbf{z}^C, \mathbf{a}^C)$, $\mathbf{D}^2\rho_{ij}^{kC}$, using ρ_{ij}^k in (3). Then using $\mathbf{D}^2\rho_{ij}^{kC}$ we find the Hessian of j's national vote share, $\mathbf{D}^2V_j$ at $(\mathbf{z}^C, \mathbf{a}^C)$.

$\mathbf{D}^2\rho_j^{kC}$ is the partial derivative of (9) wrt z_j and a_j evaluated at (z_j^C, a_j^C), i.e.,

$$\begin{aligned}\mathbf{D}^2\rho_{ij}^{kC} &\equiv \mathbf{D}^2\rho_{ij}^{k}(\mathbf{z}^C, \mathbf{a}^C) \equiv \begin{pmatrix} \dfrac{\partial^2\rho_{ij}^k}{\partial(z_j)^2} & \dfrac{\partial^2\rho_{ij}^k}{\partial z_j\partial a_j} \\ \dfrac{\partial^2\rho_{ij}^k}{\partial z_j\partial a_j} & \dfrac{\partial^2\rho_{ijk}}{\partial(a_j)^2}\end{pmatrix} \\ &= 2\rho_{ij}^{kC}(1-\rho_{ij}^{kC}) \\ &\quad\times\left[2(1-2\rho_{ij}^{kC})\boldsymbol{\iota}_k\mathbf{d}_{(\mathbf{z}^C,\mathbf{a}^C)}^{ijk\mathbf{T}}\mathbf{d}_{(\mathbf{z}^C,\mathbf{a}^C)}^{ijk}\boldsymbol{\iota}_k - \boldsymbol{\iota}_k\right] \\ &= 2\rho_{ij}^{kC}(1-\rho_{ij}^{kC})\left[2(1-2\rho_{ij}^{kC})\boldsymbol{\delta}_{ij}^{kC} - \boldsymbol{\iota}_k\right] \end{aligned} \tag{47}$$

$$\begin{aligned}&\text{where } \boldsymbol{\iota}_k \equiv \begin{pmatrix} b_k & 0 \\ 0 & e_k\end{pmatrix}, \\ &\mathbf{d}_{(\mathbf{z}^C,\mathbf{a}^C)}^{ijk} \equiv \left[(x_i - z_j^C) \quad (t_i - a_j^C)\right]\end{aligned} \tag{48}$$

and where $\mathbf{T}$ denotes the transpose of a vector. The diagonal of $\boldsymbol{\iota}_k$ gives the *importance* state k voters give to each dimension and $\mathbf{d}_{(\mathbf{z}^C,\mathbf{a}^C)}^{ijk}$ measures the *distance* between i's ideals (x_i, t_i) and j's critical choices (z_j^C, a_j^C). The matrix $\boldsymbol{\delta}_{ij}^{kC}$ in (47) defined by

$$\begin{aligned}\boldsymbol{\delta}_{ij}^{kC} &\equiv \boldsymbol{\iota}_k\mathbf{d}_{(\mathbf{z}^C,\mathbf{a}^C)}^{ijk\mathbf{T}}\mathbf{d}_{(\mathbf{z}^C,\mathbf{a}^C)}^{ijk}\boldsymbol{\iota}_k \\ &\equiv \begin{pmatrix} b_k(x_i - z_j^C)^2 b_k & b_k(x_i - z_j^C)(t_i - a_j^C)e_k \\ b_k(x_i - z_j^C)(t_i - a_j^C)e_k & e_k(t_i - a_j^C)^2 e_k\end{pmatrix}\end{aligned} \tag{49}$$

gives the *state k weighted variance–covariance matrix* of *i's ideals around j's critical choices* where the state k weights are given by $\boldsymbol{\iota}_k$ in (48).

To find if j is maximizing its *national* vote share function at $(\mathbf{z}^C, \mathbf{a}^C)$, we need the *Hessian* of second order partial derivatives of j's *national* vote share, $\mathbf{D}^2V_j(\mathbf{z}, \mathbf{a})$ evaluated at $(\mathbf{z}^C, \mathbf{a}^C)$

$$\begin{aligned}\mathbf{D}^2V_j^C &\equiv \mathbf{D}^2V_j(\mathbf{z}^C, \mathbf{a}^C) = \sum_{k\in\mathcal{S}}\frac{n_k}{n}\mathbf{D}^2 v_j^{kC} \\ &= \sum_{k\in\mathcal{S}}\frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\mathbf{D}^2\rho_{ij}^{kC} \\ &= \sum_{k\in\mathcal{S}}\frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k}2\rho_{ij}^{kC}(1-\rho_{ij}^{kC}) \\ &\quad\times\left[2(1-2\rho_{ij}^{kC})\boldsymbol{\delta}_{ij}^{kC} - \boldsymbol{\iota}_k\right]\end{aligned} \tag{50}$$

where in the last term we substitute in $\mathbf{D}^2\rho_{ij}^k$ in (47) and where $\boldsymbol{\delta}_{ij}^{kC}$ is given in (49) and j's expected state k vote share v_j^{kC} by (6).

Candidate j's national vote share, V_j, is at a maximum at $(\mathbf{z}^C, \mathbf{a}^C)$ when j's Hessian, $\mathbf{D}^2V_j^C$ in (50), is negative (semi-) definite which happens when the trace and the determinant of $\mathbf{D}^2V_j^C$ are respectively negative and positive as in this case the eigenvalues of $\mathbf{D}^2V_j^C$ are both negative. To show that $(\mathbf{z}^C, \mathbf{a}^C)$ is a *strict (weak)* local Nash equilibrium (LNE) the Hessian of all candidates, $\mathbf{D}^2V_j^C$ for all $j \in \mathcal{C}$, must be negative (semi-) definite.

Before proceeding, we give the following definitions used to simplify notation later on.

Definition 4. Let $\mathbf{c}_{ij}^{kC}$ be voter i's *characteristic matrix* for candidate j in state k at $(\mathbf{z}^C, \mathbf{a}^C)$,

$$\begin{aligned}\mathbf{c}_{ij}^{kC} &\equiv \mathbf{c}_{ij}^{k}(\mathbf{z}^C, \mathbf{a}^C) \equiv f_{ij}^{kC}\boldsymbol{\delta}_{ij}^{kC} - \boldsymbol{\iota}_k \\ &= \begin{pmatrix} f_{ij}^{kC}b_k(x_i - z_j^C)^2 b_k - b_k & f_{ij}^{kC}b_k(x_i - z_j^C)(t_i - a_j^C)e_k \\ f_{ij}^{kC}b_k(x_i - z_j^C)(t_i - a_j^C)e_k & f_{ij}^{kC}e_k(t_i - a_j^C)^2 e_k - e_k\end{pmatrix}\end{aligned} \tag{51}$$

$$\text{where } f_{ij}^{kC} \equiv 2(1-2\rho_{ij}^{kC}) \tag{52}$$

is the characteristic factor of $\mathbf{c}_{ij}^{kC}$, given $\boldsymbol{\delta}_{ij}^{kC}$ in (49), ρ_{ij}^{kC} in (16) and $\boldsymbol{\iota}_k$ in (48) and where the trace of $\mathbf{c}_{ij}^{kC}$ is given by

$$\begin{aligned}Tr(\mathbf{c}_{ij}^{kC}) &\equiv f_{ij}^{kC}Tr(\boldsymbol{\delta}_{ij}^{kC}) - Tr(\boldsymbol{\iota}_k) \\ &= f_{ij}^{kC}b_k(x_i - z_j^C)^2 b_k \\ &\quad + f_{ij}^{kC}e_k(t_i - a_j^C)^2 e_k - (b_k + e_k)\end{aligned} \tag{53}$$

$$\text{with } Tr(\boldsymbol{\iota}_k) = b_k + e_k. \tag{54}$$

Note that $Tr(\boldsymbol{\delta}_{ij}^{kC})$ and $Tr(\boldsymbol{\iota}_k)$ are also given by (34) and (32) in the text.

Denote by $\mathbf{G}_j^{kC}$ candidate j's *state k* characteristic matrix at $(\mathbf{z}^C, \mathbf{a}^C)$ given by

$$\mathbf{G}_j^{kC} \equiv \frac{1}{n_k}\sum_{i\in\mathcal{N}_k}2q_{ij}^{kC}\left[f_{ij}^{kC}\boldsymbol{\delta}_{ij}^{kC} - \boldsymbol{\iota}_k\right] = \frac{1}{n_k}\sum_{i\in\mathcal{N}_k}2q_{ij}^{kC}\mathbf{c}_{ij}^{kC} \tag{55}$$

so that after substituting in $Tr(\mathbf{c}_{ij}^{kC})$ in (53), we obtain that $Tr(\mathbf{G}_j^{kC})$ is given by

$$\begin{aligned}Tr(\mathbf{G}_j^{kC}) &= \frac{1}{n_k}\sum_{i\in\mathcal{N}_k}2q_{ij}^{kC}Tr(\mathbf{c}_{ij}^{kC}) \\ &= \frac{1}{n_k}\sum_{i\in\mathcal{N}_k}2q_{ij}^{kC}\left[f_{ij}^{kC}b_k(x_i - z_j^C)^2 b_k + f_{ij}^{kC}e_k(t_i - a_j^C)^2 e_k\right] \\ &\quad - (b_k + e_k)\frac{1}{n_k}\sum_{i\in\mathcal{N}_k}2q_{ij}^{kC}.\end{aligned} \tag{56}$$

Note that $\mathbf{G}_j^{kC}$ in (55) is the weighted average of the characteristic matrices of voters in state k, $\mathbf{c}_{ij}^{kC}$ in (51), weight by q_{ij}^{kC} in (15) and that $\mathbf{G}_j^{kC}$ varies across candidates and states. Its trace, $Tr(\mathbf{G}_j^{kC})$ in (56), is given by a weighted average of the trace of voters' characteristic matrices in state k weight by q_{ij}^{kC}.

Using $\boldsymbol{\delta}_{ij}^{kC}$ in (49) and q_{ij}^{kC} in (15), $\mathbf{D}^2V_j^C$ in (50) can be re-written as a weighted function of *voters*' characteristic matrices, $\mathbf{c}_{ij}^{kC}$ in (51) for $i \in \mathcal{N}_k$ and $k \in \mathcal{S}$, or as a weighted function of *states*' characteristic matrices, $\mathbf{G}_j^{kC}$ in (55) for $k \in \mathcal{S}$, i.e.,

$$\mathbf{D}^2V_j^C = \sum_{k\in\mathcal{S}}\frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k}2q_{ij}^{kC}\mathbf{c}_{ij}^{kC} = \sum_{k\in\mathcal{S}}\frac{n_k}{n}\mathbf{G}_j^{kC}. \tag{57}$$

Using $Tr(\mathbf{c}_{ij}^{kC})$ in (53) or $Tr(\mathbf{G}_j^{kC})$ in (56), the trace of $\mathbf{D}^2V_j^C$ in (50) is then given by

$$Tr\left(\mathbf{D}^2V_j^C\right) = \sum_{k\in\mathcal{S}} \frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k} 2q_{ij}^{kC} Tr(\mathbf{c}_{ij}^{kC}) = \sum_{k\in\mathcal{S}} \frac{n_k}{n} Tr(\mathbf{G}_j^{kC}). \tag{58}$$

From the last term in (58), we know that the sign of $Tr\left(\mathbf{D}^2V_j^C\right)$ depends on whether the $\frac{n_k}{n}$ weighted sum of $Tr(\mathbf{G}_j^{kC})$ across states is positive or negative. However, as seen shortly, the sign of $Tr(\mathbf{G}_j^{kC})$ can be positive or negative for different states. Since $\frac{n_k}{n}$ is always positive, if $Tr(\mathbf{G}_j^{kC}) < 0$ for all $k \in \mathcal{S}$ then $Tr\left(\mathbf{D}^2V_j^C\right) < 0$. Moreover, from (56), $Tr(\mathbf{G}_j^{kC})$ in state k is the q_{ij}^{kC} weighted sum of $Tr(\mathbf{c}_{ij}^{kC})$. Since q_{ij}^{kC} in (15) is positive for all $i \in \mathcal{N}_k$, $k \in \mathcal{S}$ and $j \in \mathcal{C}$ and as seen shortly $Tr(\mathbf{c}_{ij}^{kC})$ in (53) can be positive or negative for voters in state k for all $k \in \mathcal{S}$, the sign of $Tr(\mathbf{G}_j^{kC})$ in (56) depends on whether the q_{ij}^C weighted sum of $Tr(\mathbf{c}_{ij}^{kC})$ across voters in state k is positive or negative. Evidently, if $Tr(\mathbf{c}_{ij}^{kC}) < 0$ for all $i \in \mathcal{N}_k$ and $k \in \mathcal{S}$ so that $Tr(\mathbf{G}_j^{kC}) < 0$ for all $k \in \mathcal{S}$ then $Tr\left(\mathbf{D}^2V_j^C\right) < 0$. These strong conditions on voters and states are not necessary as we now show that if $Tr(\mathbf{c}_{ij}^{kC})$ in (51) is negative for *enough voters* in state k so that for state k $Tr(\mathbf{G}_j^{kC}) < 0$ and if there are *enough states* for whom $Tr(\mathbf{G}_j^{kC}) < 0$, then $Tr\left(\mathbf{D}^2V_j^C\right)$ in (50) is negative.

We now determine the conditions under which $Tr(\mathbf{c}_{ij}^{kC})$ in (53) is negative to then derive those under which $Tr(\mathbf{G}_j^{kC})$ in (56) is negative in state k, to then establish those under which $Tr\left(\mathbf{D}^2V_j^C\right)$ in (50) is negative.

From (53), $Tr(\mathbf{c}_{ij}^{kC}) < 0$ when

$$f_{ij}^{kC} < \frac{Tr(\boldsymbol{\iota}_k)}{Tr\left(\boldsymbol{\delta}_{ij}^{kC}\right)}, \tag{59}$$

so that after substituting f_{ij}^{kC} from (52) into (59) and some manipulation, we obtain

$$\frac{1}{2} - \frac{1}{4}\frac{Tr(\boldsymbol{\iota}_k)}{Tr\left(\boldsymbol{\delta}_{ij}^{kC}\right)} < \rho_{ij}^{kC} \tag{60}$$

where ρ_{ij}^{kC} is given by (3) at $(\mathbf{z}^C, \mathbf{a}^C)$. Define the LHS of (60) as voter *i's necessary pivotal probability for candidate j in state k* at $(\mathbf{z}^C, \mathbf{a}^C)$, i.e.,

$$\varphi_{ij}^{kC} \equiv \varphi_{ij}^{k}(\mathbf{z}^C, \mathbf{a}^C) \equiv \frac{1}{2} - \frac{1}{4}\frac{Tr(\boldsymbol{\iota}_k)}{Tr\left(\boldsymbol{\delta}_{ij}^{kC}\right)} = \frac{1}{2} - \frac{1}{4}\frac{b_k + e_k}{b_k(x_i - z_j^C)^2 b_k + e_k(t_i - a_j^C)^2 e_k} \tag{61}$$

where in the last term the numerator is given by (54) and the denominator[31] using (49) by

$$Tr\left(\boldsymbol{\delta}_{ij}^{kC}\right) = b_k(x_i - z_j^C)^2 b_k + e_k(t_i - a_j^C)^2 e_k. \tag{62}$$

Then using (61), the *necessary condition* on voter i in state k for j to adopt (z_j^C, a_j^C) as its campaign given in (60) becomes

$$\varphi_{ij}^{kC} < \rho_{ij}^{kC}. \tag{63}$$

[31] We assume that no voter's policy and campaign tolerance are at z_j^C and a_j^C to avoid dividing by zero.

This condition says that if the probability that i votes for j in state k at $(\mathbf{z}^C, \mathbf{a}^C)$, ρ_{ij}^{kC}, is *higher* than i's necessary pivotal probability for j in state k at $(\mathbf{z}^C, \mathbf{a}^C)$, φ_{ij}^{kC}, then $Tr(\mathbf{c}_{ij}^{kC}) < 0$. For *some* voters $i \in \mathcal{N}_k$, the probability of voting for j may be *too low* in the sense that $\varphi_{ij}^{k} > \rho_{ij}^{kC}$, for these voters $Tr(\mathbf{c}_{ij}^{kC}) > 0$.

Having determined the condition under which $Tr(\mathbf{c}_{ij}^{kC}) < 0$ for voters in state k, we now study the conditions under which $Tr(\mathbf{G}_j^{kC})$ in (56) in state k is negative. Aggregating the left hand side (LHS) of (63) over voters in state k and dividing by n_k gives *j's necessary state k pivotal vote share* at $(\mathbf{z}^C, \mathbf{a}^C)$, i.e.,

$$\Phi_j^{kC} \equiv \Phi_j^{k}(\mathbf{z}^C, \mathbf{a}^C) = \frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\varphi_{ij}^{kC} \equiv \frac{1}{2} - \frac{1}{4}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\frac{Tr(\boldsymbol{\iota}_k)}{Tr\left(\boldsymbol{\delta}_{ij}^{kC}\right)} = \frac{1}{2} - \frac{1}{4}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\frac{b_k + e_k}{b_k(x_i - z_j^C)^2 b_k + e_k(t_i - a_j^C)^2 e_k} \tag{64}$$

where the last term follows from (61). Note that Φ_j^{kC} in (64) varies across candidates and states as it depends on (z_j^C, a_j^C), n_k and the distribution of voters' preferences.

Averaging the right hand side (RHS) of (63) over voters in state k, i.e., over $\mathcal{N}_k$, gives *j's expected state k vote share* at $(\mathbf{z}^C, \mathbf{a}^C)$, υ_j^{kC} in (6).

If (63) is satisfied for all $i \in \mathcal{N}_k$, i.e., if $\varphi_{ij}^{kC} < \rho_{ij}^{kC}$ for all $i \in \mathcal{N}_k$, then $\Phi_j^{kC} < \upsilon_j^{kC}$ making the trace of state k's characteristic matrix, $Tr(\mathbf{G}_j^{kC})$ in (56), negative.

We now show that for $Tr(\mathbf{G}_j^{kC})$ to be negative we do *not* require that all voters in state k vote for j with high enough probability, that is, we do not need that $\varphi_{ij}^{kC} < \rho_{ij}^{kC}$ for all $i \in \mathcal{N}_k$. To show this we partition the set of *voters in state k*, $\mathcal{N}_k$, into three subsets according to whether $Tr(\mathbf{c}_{ij}^{kC}) \gtreqqless 0$. Let $\mathcal{N}_k^{T-}$ be the set of voters in state k for whom $Tr(\mathbf{c}_{ij}^{kC}) < 0$, i.e., for whom $\varphi_{ij}^{kC} < \rho_{ij}^{kC}$; $\mathcal{N}_k^{T+}$ those for whom $Tr(\mathbf{C}_{ij}^{kC}) > 0$, i.e., for whom $\varphi_{ij}^{kC} > \rho_{ij}^{kC}$; and $\mathcal{N}_k^{T0}$ those for whom $Tr(\mathbf{c}_{ij}^{kC}) = 0$, i.e., for whom $\varphi_{ij}^{kC} = \rho_{ij}^{kC}$.

Since q_{ij}^{kC} in (17) is positive for all $i \in \mathcal{N}_k$, the sign of $2q_{ij}^{kC}Tr(\mathbf{c}_{ij}^{kC})$ depends on the sign of $Tr(\mathbf{c}_{ij}^{kC})$. Thus, when aggregating $2q_{ij}^{kC}Tr(\mathbf{C}_{ij}^{kC})$ over $\mathcal{N}_k^{T-}$, $\mathcal{N}_k^{T+}$ and $\mathcal{N}_k^{T0}$, we get that

$$\sum_{i\in\mathcal{N}_k^{T-}} 2q_{ij}^{kC}Tr(\mathbf{c}_{ij}^{kC}) < 0, \qquad \sum_{i\in\mathcal{N}_k^{T+}} 2q_{ij}^{kC}Tr(\mathbf{c}_{ij}^{kC}) > 0 \quad \text{and} \quad \sum_{i\in\mathcal{N}_k^{T0}} 2q_{ij}^{kC}Tr(\mathbf{c}_{ij}^{kC}) = 0.$$

So that $Tr(\mathbf{G}_j^{kC})$ in (56) can be re-written as

$$Tr(\mathbf{G}_j^{kC}) = \frac{1}{n_k}\sum_{i\in\mathcal{N}_k} 2q_{ij}^{kC}Tr(\mathbf{c}_{ij}^{kC}) = \frac{1}{n_k}\left[\sum_{i\in\mathcal{N}_k^{T-}} 2q_{ij}^{kC}Tr(\mathbf{c}_{ij}^{kC}) + \sum_{i\in\mathcal{N}_k^{T+}} 2q_{ij}^{kC}Tr(\mathbf{c}_{ij}^{kC})\right].$$

Clearly, $Tr(\mathbf{G}_j^{kC}) < 0$ iff $\left|\sum_{i\in\mathcal{N}_k^{T-}} q_{ij}^{kC}Tr(\mathbf{c}_{ij}^{kC})\right| > \left|\sum_{i\in\mathcal{N}_k^{T+}} q_{ij}^{kC}Tr(\mathbf{c}_{ij}^{kC})\right|$. Thus, if $Tr(\mathbf{c}_{ij}^{kC}) < 0$ for *enough* voters in state k, i.e., if $\varphi_{ij}^{kC} < \rho_{ij}^{kC}$ in (63) is satisfied for *enough* voters in *state* k then $Tr(\mathbf{G}_j^{kC}) < 0$. This says that the probability that voters in state k vote for j must be high enough for enough voters in state k for j to adopt (z_j^C, a_j^C) as its campaign in state k.

Summarizing, if $\varphi_{ij}^{kC} < \rho_{ij}^{kC}$ in (63) is satisfied for *enough* voters in *state* k then $Tr(\mathbf{G}_j^{kC}) < 0$. Thus, the *state k necessary condition* for j to adopt (z_j^C, a_j^C) as its campaign in *state k* – holding $(\mathbf{z}_{-j}^C, \mathbf{a}_{-j}^C)$ constant – is that

$$\Phi_j^{kC} < v_j^{kC}. \tag{65}$$

This condition says that j's expected state k vote share, v_j^{kC} in (6), must be *high* enough at $(\mathbf{z}^C, \mathbf{a}^C)$, i.e., higher than j's necessary pivotal vote share in state k, Φ_j^{kC} in (64), for j to adopt (z_j^C, a_j^C) as its campaign in state k.

Having determined the conditions under which $Tr(\mathbf{G}_j^{kC}) < 0$ in state k, we now examine the conditions under which $Tr\left(\mathbf{D}^2 V_j^C\right) = \sum_{k\in\mathcal{S}} \frac{n_k}{n} Tr(\mathbf{G}_j^{kC})$ in (58) is negative.

Multiply the LHS of (65) by state k's share of the national population, $\frac{n_k}{n}$, and sum over all states to obtain j's *necessary national pivotal vote share*, ϑ_j^C at $(\mathbf{z}^C, \mathbf{a}^C)$,

$$\vartheta_j^C \equiv \vartheta_j(\mathbf{z}^C, \mathbf{a}^C) \equiv \sum_{k\in\mathcal{S}} \frac{n_k}{n} \Phi_{ij}^{kC} \equiv \sum_{k\in\mathcal{S}} \frac{n_k}{n} \frac{1}{n_k} \sum_{i\in\mathcal{N}_k} \varphi_{ij}^{kC}$$

$$\equiv \frac{1}{2} - \frac{1}{4} \sum_{k\in\mathcal{S}} \frac{n_k}{n} \frac{1}{n_k} \sum_{i\in\mathcal{N}_k} \frac{Tr(\boldsymbol{\iota}_k)}{Tr\left(\boldsymbol{\delta}_{ij}^{kC}\right)} \tag{66}$$

where $Tr\left(\boldsymbol{\delta}_{ij}^{kC}\right)$ is given by (62) and $Tr(\boldsymbol{\iota}_k)$ by (54), and where the last term follows after substituting Φ_{ij}^{kC} in (64) or φ_{ij}^{kC} in (61). Since $Tr\left(\boldsymbol{\delta}_{ij}^{kC}\right)$ depends on (z_j^C, a_j^C), so does j's necessary national pivotal vote share, implying that ϑ_j^C varies across candidates.

Similarly, when multiplying the RHS of (65) by state k's share of the national population, $\frac{n_k}{n}$, and summing over all states gives j's *national expected vote share*, V_j^C in (7) or in (5).

If $Tr(\mathbf{G}_j^{kC})$ in (56) is negative for all $k \in \mathcal{S}$ then from (65) $\Phi_j^{kC} < v_j^{kC}$ for all $k \in \mathcal{S}$, so that j's *national* vote share at $(\mathbf{z}^C, \mathbf{a}^C)$, V_j^C in (7), is greater than j's national pivotal vote share, ϑ_j^C in (66), i.e., $\vartheta_j^C < V_j^C$, implying that $Tr\left(\mathbf{D}^2 V_j^C\right) < 0$. However, for some states, $Tr(\mathbf{G}_j^{kC})$ may be positive, so that in these states the probability that state k votes for j is *too low*, i.e., $\Phi_j^{kC} > v_j^{kC}$.

We now show that we do not need $Tr(\mathbf{G}_j^{kC})$ in (56) to be negative for all $k \in \mathcal{S}$ for $Tr\left(\mathbf{D}^2 V_j^C\right) < 0$. To do so, partition the set of *states* $\mathcal{S}$ into three subsets. Let $\mathcal{S}^{T-}$ be the set of states for whom $Tr(\mathbf{G}_j^{kC}) < 0$, i.e., for whom $\Phi_j^{kC} < v_{ij}^{kC}$; $\mathcal{S}^{T+}$ those for whom $Tr(\mathbf{G}_j^{kC}) > 0$, i.e., those for whom $\Phi_j^{kC} > v_{ij}^{kC}$; and $\mathcal{S}^{T0}$ those for whom $Tr(\mathbf{G}_j^{kC}) = 0$ i.e., for whom $\Phi_j^{kC} = v_j^{kC}$.

Given that state k's share of the national population $\frac{n_k}{n}$ is positive for all $k \in \mathcal{S}$, the sign of $\frac{n_k}{n} Tr(\mathbf{G}_j^{kC})$ depends only on the sign of $Tr(\mathbf{G}_j^{kC})$. Thus, when aggregating $\frac{n_k}{n} Tr(\mathbf{G}_j^{kC})$ over $\mathcal{S}^{T-}$, $\mathcal{S}^{T+}$ and $\mathcal{S}^{T0}$, we get that

$$\sum_{k\in\mathcal{S}^{T-}} \frac{n_k}{n} Tr(\mathbf{G}_j^{kC}) < 0, \qquad \sum_{k\in\mathcal{S}^{T+}} \frac{n_k}{n} Tr(\mathbf{G}_j^{kC}) > 0 \quad \text{and}$$
$$\sum_{k\in\mathcal{S}^{T0}} \frac{n_k}{n} Tr(\mathbf{G}_j^{kC}) = 0.$$

The trace of $\mathbf{D}^2 V_j^C$ in (58) can be decomposed into

$$Tr\left(\mathbf{D}^2 V_j^C\right) = \sum_{k\in\mathcal{S}} \frac{n_k}{n} Tr(\mathbf{G}_j^{kC})$$
$$= \sum_{k\in\mathcal{S}^{T-}} \frac{n_k}{n} Tr(\mathbf{G}_j^{kC}) + \sum_{k\in\mathcal{S}^{T+}} \frac{n_k}{n} Tr(\mathbf{G}_j^{kC}).$$

Thus, $Tr\left(\mathbf{D}^2 V_j^C\right) < 0$ iff $\left|\sum_{k\in\mathcal{S}^{T-}} \frac{n_k}{n} Tr(\mathbf{G}_j^{kC})\right| > \left|\sum_{k\in\mathcal{S}^{T+}} \frac{n_k}{n} Tr(\mathbf{G}_j^{kC})\right|$. Therefore, if $Tr(\mathbf{G}_j^{kC}) < 0$ for *enough* states, i.e., if $\Phi_j^{kC} < v_j^{kC}$ in (65) is satisfied for *enough states* then $Tr\left(\mathbf{D}^2 V_j^C\right) < 0$. This says that if there are enough states from whom the probability of voting for j is high enough, i.e., if j's vote share v_j^{kC} is high enough, i.e., higher than Φ_j^{kC} for enough states then $Tr\left(\mathbf{D}^2 V_j^C\right) < 0$.

Recall that $Tr(\mathbf{G}_j^{kC}) < 0$ only if (63) is satisfied for enough voters in these states. Thus, the probability that states vote for candidate j must be high enough in enough states and this happens only if there are enough voters in these states for whom the probability of voting for j is high enough. If $\Phi_j^{kC} < v_j^{kC}$ in (65) is satisfied for enough states then $Tr\left(\mathbf{D}^2 V_j^C\right) < 0$.

Therefore, the *necessary condition* for j to adopt (z_j^C, a_j^C) as its electoral campaign – when $(\mathbf{z}_{-j}^C, \mathbf{a}_{-j}^C)$ is held constant – is that

$$\vartheta_j^C < V_j^C. \tag{67}$$

This condition says that when j's national vote share at $(\mathbf{z}^C, \mathbf{a}^C)$, V_j^C, is *higher* than j's necessary pivotal vote share at $(\mathbf{z}^C, \mathbf{a}^C)$, ϑ_j^C, then $Tr\left(\mathbf{D}^2 V_j^C\right) < 0$.

When candidate j's national vote share is too low, i.e., when $\vartheta_j^C > V_j^C$, j does *not* adopt (z_j^C, a_j^C) as its electoral campaign as j's expected national vote share is at a *minimum or at a saddle point* at (z_j^C, a_j^C). By changing its policy and/or ad campaign j can increase its expected national vote share. When j changes its electoral campaign, other candidates may also find it in their interest to also change their national campaign. In this case, candidates' critical campaigns $(\mathbf{z}^C, \mathbf{a}^C)$ are *not* a LNE of the election.

Before examining the sufficient conditions for candidates to adopt $(\mathbf{z}^C, \mathbf{a}^C)$ as their electoral campaign, we give the following definitions that help simply notation later on.

Definition 5. Using q_{ij}^{kC} in (15), let q_j^{kC} denote the average probability that state k votes for j and for any other candidate at $(\mathbf{z}^C, \mathbf{a}^C)$,

$$q_j^{kC} \equiv \frac{1}{n_k} \sum_{i\in\mathcal{N}_k} q_{ij}^{kC} = \frac{1}{n_k} \sum_{i\in\mathcal{N}_k} \rho_{ij}^{kC}(1-\rho_{ij}^{kC}). \tag{68}$$

Let the matrix $\boldsymbol{\Gamma}_j^C$ denote the weighted aggregate national importance that candidate j gives to the two dimensions at $(\mathbf{z}^C, \mathbf{a}^C)$,

$$\boldsymbol{\Gamma}_j^C \equiv \sum_{k\in\mathcal{S}} \frac{n_k}{n} \boldsymbol{\iota}_k q_j^{kC} = \sum_{k\in\mathcal{S}} \frac{n_k}{n} \begin{pmatrix} b_k q_j^{kC} & 0 \\ 0 & e_k q_j^{kC} \end{pmatrix} \tag{69}$$

$$= \sum_{k\in\mathcal{S}} \frac{n_k}{n} \frac{1}{n_k} \sum_{i\in\mathcal{N}_k} \begin{pmatrix} b_k \rho_{ij}^{kC}(1-\rho_{ij}^{kC}) & 0 \\ 0 & e_k \rho_{ij}^{kC}(1-\rho_{ij}^{kC}) \end{pmatrix} \tag{70}$$

$$= \begin{pmatrix} \boldsymbol{\Gamma}_j^C(z) \equiv \frac{1}{n} \sum_{k\in\mathcal{S}} \sum_{i\in\mathcal{N}_k} b_k \rho_{ij}^{kC}(1-\rho_{ij}^{kC}) & 0 \\ 0 & \boldsymbol{\Gamma}_j^C(a) \equiv \frac{1}{n} \sum_{k\in\mathcal{S}} \sum_{i\in\mathcal{N}_k} e_k \rho_{ij}^{kC}(1-\rho_{ij}^{kC}) \end{pmatrix} \tag{71}$$

which depends on the importance voters give to the national dimensions in state k, $\boldsymbol{\iota}_k$, weight by the state's proportion of the national population, $\frac{n_k}{n}$ and by the average probability that state k votes for j and for any other candidate, q_j^{kC} in (68). Note that using (68), the determinant of $\boldsymbol{\Gamma}_j^C$ is always positive, i.e.,

$$\det(\boldsymbol{\Gamma}_j^C) = \left[\sum_{k\in\mathcal{S}} \frac{n_k}{n} b_k q_j^{kC}\right] \left[\sum_{k\in\mathcal{S}} \frac{n_k}{n} e_k q_j^{kC}\right]$$
$$= \left[\sum_{k\in\mathcal{S}} \frac{n_k}{n} b_k \rho_{ij}^{kC}(1-\rho_{ij}^{kC})\right]$$
$$\times \left[\sum_{k\in\mathcal{S}} \frac{n_k}{n} e_k \rho_{ij}^{kC}(1-\rho_{ij}^{kC})\right] > 0. \tag{72}$$

Given q_{ij}^{kC} in (15), f_{ij}^{kC} in (52) and δ_{ij}^{kC} in (49), let $\mathbf{R}_j^C \equiv \mathbf{R}_j(\mathbf{z}^C, \mathbf{a}^C)$ represent the following matrix at $(\mathbf{z}^C, \mathbf{a}^C)$

$$\mathbf{R}_j^C \equiv \mathbf{R}_j(\mathbf{z}^C, \mathbf{a}^C) \equiv \sum_{k\in\mathcal{S}} \frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k} 2q_{ij}^{kC} f_{ij}^{kC}\delta_{ij}^{kC} \tag{73}$$

whose determinant is given by

$$\begin{aligned}\det\left[\mathbf{R}_j^C\right] = &\left[\sum_{k\in\mathcal{S}} \frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k} 2q_{ij}^{kC} f_{ij}^{kC} b_k(x_i - z_j^C)^2 b_k\right] \\ &\times\left[\sum_{k\in\mathcal{S}} \frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k} 2q_{ij}^{kC} f_{ij}^{kC} e_k(t_i - a_j^C)^2 e_k\right] \\ &-\left[\sum_{k\in\mathcal{S}} \frac{n_k}{n}\frac{1}{n_k}\right. \\ &\left.\times\sum_{i\in\mathcal{N}_k} 2q_{ij}^{kC} f_{ij}^{kC} b_k(x_i - z_j^C)(t_i - a_j^C) e_k\right]^2.\end{aligned} \tag{74}$$

The *sufficient SOC* for candidate j to adopt (z_j^C, a_j^C) as its campaign, given $(\mathbf{z}_{-j}^C, \mathbf{a}_{-j}^C)$, is that the eigenvalues of $\mathbf{D}^2 V_j$ in (50) evaluated at $(\mathbf{z}^C, \mathbf{a}^C)$, $\mathbf{D}^2 V_j^C$, be *both* negative implying that the *determinant* of $\mathbf{D}^2 V_j$ must be positive at $(\mathbf{z}^C, \mathbf{a}^C)$.

From (57), the determinant of $\mathbf{D}^2 V_j$ at $(\mathbf{z}^C, \mathbf{a}^C)$ is given by

$$\begin{aligned}\det\left(\mathbf{D}^2 V_j^C\right) &\equiv \det\left(\mathbf{D}^2 V_j(\mathbf{z}^C, \mathbf{a}^C)\right) \\ &= \det\left(\sum_{k\in\mathcal{S}} \frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k} 2q_{ij}^{kC}\left[f_{ij}^{kC}\delta_{ij}^{kC} - \boldsymbol{\iota}_k\right]\right)\end{aligned}$$

where f_{ij}^{kC} is given by (52), q_{ij}^{kC} by (15), δ_{ij}^{kC} by (49) and $\boldsymbol{\iota}_k$ by (48). After some manipulation and using $\det(\mathbf{R}_j^C)$ in (74) and $Tr(\boldsymbol{\iota}_k)$ in (54), $\det(\mathbf{D}^2 V_j^C)$ can be re-arranged into components that depend on i's variance–covariance matrix around j's critical choices given in $\det\left[\mathbf{R}_j^C\right]$ in (74) and components that also depend on the importance voters give to the two dimensions in state k, i.e., that also depend on Γ_j^C in (69), i.e.,

$$\begin{aligned}&\det(\mathbf{D}^2 V_j^C) \\ &= \det\left[\mathbf{R}_j^C\right] + 4\left[\sum_{k\in\mathcal{S}} \frac{n_k}{n} b_k q_j^{kC}\right]\left[\sum_{k\in\mathcal{S}} \frac{n_k}{n} e_k q_j^{kC}\right] \\ &\quad\times\left\{1 - \sum_{k\in\mathcal{S}} \frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k} q_{ij}^{kC} f_{ij}^{kC}\, Tr\left((\Gamma_j^C)^{-1}\delta_{ij}^{kC}\right)\right\}\end{aligned} \tag{75}$$

where $(\Gamma_j^C)^{-1}$ is the inverse of Γ_j^C in (69) and $Tr\left((\Gamma_j^C)^{-1}\delta_{ij}^{kC}\right)$ is given by

$$\begin{aligned}Tr\left((\Gamma_j^C)^{-1}\delta_{ij}^{kC}\right) \equiv &\frac{1}{\sum_{k\in\mathcal{S}} \frac{n_k}{n} b_k q_j^{kC}} b_k(x_i - z_j^C)^2 b_k \\ &+ \frac{1}{\sum_{k\in\mathcal{S}} \frac{n_k}{n} e_k q_j^{kC}} e_k(t_i - a_j^C)^2 e_k\end{aligned}$$

and so depends on ρ_{ij}^{kC} through q_j^{kC} in (68).

Using $\det(\Gamma_j^C)$ in (72) and since $1 = \sum_{k\in\mathcal{S}} \frac{n_k}{n}\sum_{i\in\mathcal{N}_k}\frac{1}{n_k}$, re-write $\det(\mathbf{D}^2 V_j^C)$ in (75) as

$$\begin{aligned}&\det(\mathbf{D}^2 V_j^C) \\ &\quad= \det\left[\mathbf{R}_j^C\right] + 4\det(\Gamma_j^C) \\ &\quad\times\left\{\sum_{k\in\mathcal{S}} \frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\left[1 - q_{ij}^{kC} f_{ij}^{kC}\, Tr\left((\Gamma_j^C)^{-1}\delta_{ij}^{kC}\right)\right]\right\}.\end{aligned} \tag{76}$$

Recall that the sufficient condition requires that $\det(\mathbf{D}^2 V_j^C) > 0$. Before finding the sign of $\det(\mathbf{D}^2 V_j^C)$, we need to find the sign of $\det(\mathbf{R}_j^C)$. From (74) we can see that the combined weight $q_{ij}^{kC} f_{ij}^{kC}$ appears in all three terms in square brackets in $\det(\mathbf{R}_j^C)$. Given that $q_{ij}^{kC} \equiv \rho_{ij}^{kC}(1-\rho_{ij}^{kC})$ in (15) and $f_{ij}^{kC} \equiv 2(1-2\rho_{ij}^{kC})$ in (52) depend ρ_{ij}^{kC}, the combined weight $q_{ij}^{kC} f_{ij}^{kC}$ is a cubic function of ρ_{ij}^{kC} (see Fig. 6), i.e.,

$$q_{ij}^{kC} f_{ij}^{kC} \equiv \rho_{ij}^{kC}(1-\rho_{ij}^{kC})2(1-2\rho_{ij}^{kC}) \tag{77}$$

$$\text{with a maximum at } q_{ij}^{kC} \approx 0.2113 \text{ and } (q_{ij}^{kC} f_{ij}^{kC})_{\max} \approx 0.1924 \tag{78}$$

$$\text{and a minimum at } q_{ij}^{kC} \approx 0.7887 \text{ and } (q_{ij}^{kC} f_{ij}^{kC})_{\min} \approx -0.1924. \tag{79}$$

Since it is difficult to sign $\det\left[\mathbf{R}_j^C\right]$, we use the following definitions to determine the upper and lower bounds of $\det\left[\mathbf{R}_j^C\right]$.

Definition 6. Let $d_j^M(z_j^C)$ and $d_j^m(z_j^C)$ represent the maximum (M) and minimum (m) distances between the ideal policy of any voter in the country and j's critical policy, i.e.,

$$d_j^M(z_j^C) \equiv \max\{|x_i - z_j^C| \text{ for all } i \in \mathcal{N}_k \text{ and } k \in \mathcal{S}\} \tag{80}$$

$$d_j^m(z_j^C) \equiv \min\{|x_i - z_j^C| \text{ for all } i \in \mathcal{N}_k \text{ and } k \in \mathcal{S}\} \tag{81}$$

and $d_j^M(a_j^C)$ and $d_j^m(a_j^C)$ the maximum (M) and minimum (m) distances between the campaign tolerance level of any voter in the country and j's critical advertising campaign, i.e.,

$$d_j^M(a_j^C) \equiv \max\{|t_i - a_j^C| \text{ for all } i \in \mathcal{N}_k \text{ and } k \in \mathcal{S}\} \tag{82}$$

$$d_j^m(a_j^C) \equiv \min\{|t_i - a_j^C| \text{ for all } i \in \mathcal{N}_k \text{ and } k \in \mathcal{S}\}. \tag{83}$$

Since $d_j^M(z_j^C) \geqslant d_j^m(z_j^C)$ and $d_j^M(a_j^C) \geqslant d_j^m(a_j^C)$, then $\frac{d_j^M(z_j^C)}{d_j^m(z_j^C)} \geqslant 1$ and $\frac{d_j^M(a_j^C)}{d_j^m(a_j^C)} \geqslant 1$.

Let β^M and ε^M denote the maximum (M) importance voters in any state give to the policy and advertising dimensions, i.e.,

$$\begin{aligned}\beta^M &\equiv \max\{b_k \text{ for all } k \in \mathcal{S}\} \quad \text{and} \\ \varepsilon^M &\equiv \max\{e_k \text{ for all } k \in \mathcal{S}\}\end{aligned} \tag{84}$$

and β^m and ε^m the corresponding minimum (m) importance, i.e.,

$$\begin{aligned}\beta^m &\equiv \min\{b_k \text{ for all } k \in \mathcal{S}\} \quad \text{and} \\ \varepsilon^m &\equiv \min\{e_k \text{ for all } k \in \mathcal{S}\}.\end{aligned} \tag{85}$$

So that $\beta^M \geqslant \beta^m$ and $\varepsilon^M \geqslant \varepsilon^m$ implying that $\frac{\beta^M}{\beta^m} \geqslant 1$ and $\frac{\varepsilon^M}{\varepsilon^m} \geqslant 1$.

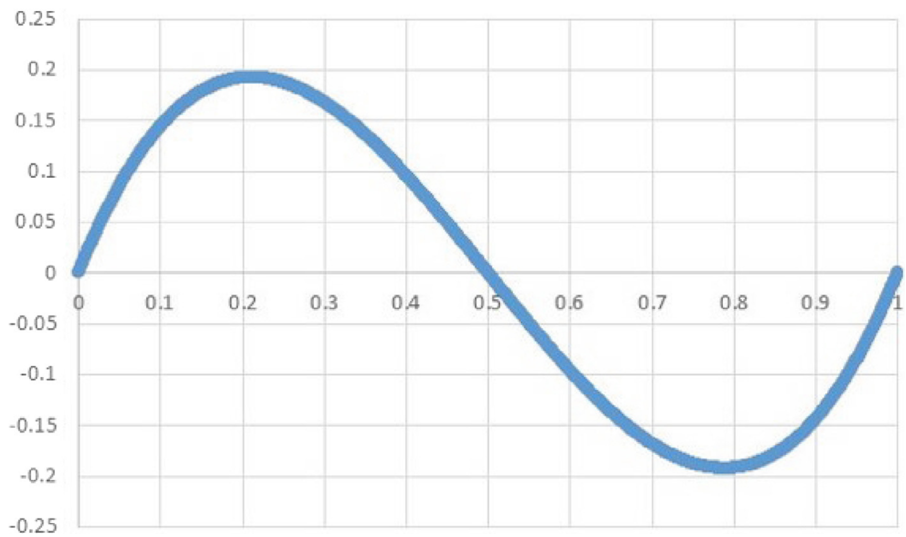

Fig. 6. Graph of $q_{ij}^{kC} f_{ij}^{kC} \equiv \rho_{ij}^{kC}(1-\rho_{ij}^{kC})2(1-2\rho_{ij}^{kC})$.

We find the upper (lower) bound of det $\left[\mathbf{R}_j^C\right]$ in several steps: determine the upper and lower bounds of each term in det $\left[\mathbf{R}_j^C\right]$ in (74) then using these upper and lower bounds find the upper and lower bounds of det $\left[\mathbf{R}_j^C\right]$.

The upper bound of the first bracketed term in (74), can be found using the maximum of $q_{ij}^{kC} f_{ij}^{kC}$ in (78), the maximum distance of voters' ideal policies from j's policy $d_j^M(z_j^C)$ in (80) and the maximum importance voters give to the policy dimension, β^M in (84). That is,

$$\sum_{k\in\mathcal{S}} \frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k} 2q_{ij}^{kC} f_{ij}^{kC} b_k(x_i - z_j^C)^2 b_k \leqslant \sum_{k\in\mathcal{S}} \frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k} 2(0.1924)\beta^M (d_j^M(z_j^C))^2\beta^M. \tag{86}$$

Since β^M in (84) and $d_j^M(z_j^C)$ in (80) do not depend on voters' policy or advertising preferences or on the importance voters in any state give to the policy dimension and since $\sum_{k\in\mathcal{S}} \frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k} 1 = 1$, the upper bound of (86) is then given by

$$\sum_{k\in\mathcal{S}} \frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k} 2q_{ij}^{kC} f_{ij}^{kC} b_k(x_i - z_j^C)^2 b_k \leqslant 2(0.1924)\beta^M (d_j^M(z_j^C))^2\beta^M. \tag{87}$$

Using ε^M in (84), $d_j^M(a_j^C)$ in (82) and maximum of $q_{ij}^{kC} f_{ij}^{kC}$ in (78), the upper bound of the second term in (74) is given by

$$\sum_{k\in\mathcal{S}} \frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k} 2q_{ij}^{kC} f_{ij}^{kC} e_k(t_i - a_j^C)^2 e_k \leqslant 2(0.1924)\varepsilon^M (d_j^M(a_j^C))^2\varepsilon^M. \tag{88}$$

Given the minimum of $q_{ij}^{kC} f_{ij}^{kC}$ in (79), β^m and ε^m in (85), $d_j^m(z_j^C)$ in (81) and $d_j^m(a_j^C)$ in (83), the *lower bound* of the last bracketed term in (74) is given by

$$\left[\sum_{k\in\mathcal{S}} \frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k} 2q_{ij}^{kC} f_{ij}^{kC} b_k(x_i - z_j^C)(t_i - a_j^C)e_k\right]^2 \geqslant \left[2(-0.1924)\beta^m d_j^m(z_j^C) d_j^m(a_j^C)\varepsilon^m\right]^2. \tag{89}$$

Combining the upper bounds of the first two components of det $\left[\mathbf{R}_j^C\right]$ in (87) and (88) and the lower bound of (89), the upper bound of det $\left[\mathbf{R}_j^C\right]$ in (74) is given by

$$\det\left[\mathbf{R}_j^C\right] \leqslant \left[2(0.1924)\beta^M(d_j^M(z_j^C))^2\beta^M\right] \times \left[2(0.1924)\varepsilon^M(d_j^M(a_j^C))^2\varepsilon^M\right] - \left[2(-0.1924)\beta^m d_j^m(z_j^C) d_j^m(a_j^C)\varepsilon^m\right]^2. \tag{90}$$

The upper bound of det $\left[\mathbf{R}_j^C\right]$ is non-negative when the RHS of (90) is non-negative which is always the case since RHS of (90) reduces to

$$\left(\frac{\beta^M}{\beta^m}\frac{\varepsilon^M}{\varepsilon^m}\right)^2 \geqslant 1 \geqslant \left(\frac{d_j^m(z_j^C) d_j^m(a_j^C)}{d_j^M(z_j^C) d_j^M(a_j^C)}\right)^2 \tag{91}$$

which is always satisfied. Thus, the RHS of (90) is always non-negative, implying that the upper bound of det $\left[\mathbf{R}_j^C\right]$ is non-negative.

Using a similar procedure and given the minimum of $q_{ij}^{kC} f_{ij}^{kC}$ in (79), β^m in (85), ε^m in (85), $d_j^m(z_j^C)$ in (81) and $d_j^m(a_j^C)$ in (83) and the maximum of $q_{ij}^{kC} f_{ij}^{kC}$ in (78), β^m and ε^M in (84), $d_j^M(z_j^C)$ in (80) and $d_j^M(a_j^C)$ in (82), the lower bound of det $\left[\mathbf{R}_j^C\right]$ is given by

$$\det\left[\mathbf{R}_j^C\right] \geqslant \left[2(-0.1924)\beta^m(d_j^m(z_j^C))^2\beta^m\right] \times \left[2(-0.1924)\varepsilon^m(d_j^m(a_j^C))^2\varepsilon^m\right] - \left[2(0.1924)\beta^M d_j^M(z_j^C) d_j^M(a_j^C)\varepsilon^M\right]^2. \tag{92}$$

The lower bound of det $\left[\mathbf{R}_j^C\right]$ is non-positive when the RHS of (92) is non-positive which is always the case since RHS of (92) reduces to

$$\left(\frac{\beta^M}{\beta^m}\frac{\varepsilon^M}{\varepsilon^m}\right)^2 \geqslant 1 \geqslant \left(\frac{d_j^m(z_j^C) d_j^m(a_j^C)}{d_j^M(z_j^C) d_j^M(a_j^C)}\right)^2 \tag{93}$$

which is always satisfied. Thus, the RHS of (90) is always non-positive, implying that the lower bound of det $\left[\mathbf{R}_j^C\right]$ is non-positive.

Thus, since condition (91) implies that the upper bound of det $\left[\mathbf{R}_j^C\right]$ is non-negative and (93) that the lower bound of det $\left[\mathbf{R}_j^C\right]$ is non-positive, and since condition (91) is exactly the same as condition (93), it must the case that det $\left[\mathbf{R}_j^C\right] = 0$.

Before determining the sign of det$(\mathbf{D}^2 V_j^C)$ in (76), we use the following definitions to simplify notation later on.

Definition 7. Let $\mathcal{A}_{ij}^{kC}$ be the short hand expression for the term in the square bracket inside the curly bracket in $\mathbf{D}^2V_j^C$ in (76), i.e.,

$$\mathcal{A}_{ij}^{kC} \equiv 1 - q_{ij}^{kC} f_{ij}^{kC} Tr\left((\mathbf{\Gamma}_j^C)^{-1}\boldsymbol{\delta}_{ij}^{kC}\right) \tag{94}$$

and let $\mathcal{B}_{ij}^{kC}$ denote the trace of $(\mathbf{\Gamma}_j^C)^{-1}\boldsymbol{\delta}_{ij}^{kC}$, i.e.,

$$\mathcal{B}_{ij}^{kC} \equiv Tr\left((\mathbf{\Gamma}_j^C)^{-1}\boldsymbol{\delta}_{ij}^{kC}\right). \tag{95}$$

Since $\det\left[\mathbf{R}_j^C\right] = 0$, the determinant of $\mathbf{D}^2V_j^C$ in (76) is positive iff the last term is positive. From (72) we know that $\det(\Gamma_j^C) > 0$ and since $\frac{n_k}{n}\frac{1}{n_k} > 0$, the determinant of $\mathbf{D}^2V_j^C$ is positive iff $\mathcal{A}_{ij}^{kC}$ in (94) is positive. Clearly, if $\mathcal{A}_{ij}^{kC} > 0$ for all $i \in \mathcal{N}_k$ and $k \in \mathcal{S}$ then $\det(\mathbf{D}^2V_j^C) > 0$. We now show that this strong condition is not required for $\det(\mathbf{D}^2V_j^C) > 0$, as we only need $\mathcal{A}_{ij}^{kC}$ to be positive for *enough voters in state k* in *enough states* for $\det(\mathbf{D}^2V_j^C) > 0$.

The term $\mathcal{A}_{ij}^{kC}$ in (94) is positive when

$$q_{ij}^{kC} f_{ij}^{kC} \equiv \rho_{ij}^{kC}(1-\rho_{ij}^{kC})2(1-2\rho_{ij}^{kC}) < \frac{1}{Tr\left((\mathbf{\Gamma}_j^C)^{-1}\boldsymbol{\delta}_{ij}^{kC}\right)}. \tag{96}$$

The cubic equation on the LHS of (96) is the same as that given in (77) and shown in Fig. 6. The RHS also depends on ρ_{ij}^{kC} though $\mathbf{\Gamma}_j^C$ in (71). To get an idea of whether $\left(Tr\left((\mathbf{\Gamma}_j^C)^{-1}\boldsymbol{\delta}_{ij}^{kC}\right)\right)^{-1} = (\mathcal{B}_{ij}^{kC})^{-1}$ – where $\mathcal{B}_{ij}^{kC}$ is given in (95) – increases or decreases in ρ_{ij}^{kC} take the derivative of $(\mathcal{B}_{ij}^{kC})^{-1}$ with respect to ρ_{ij}^{kC} keeping every thing else constant, i.e.,

$$\begin{aligned}\frac{\partial[(\mathcal{B}_{ij}^{kC})^{-1}]}{\partial\rho_{ij}^{kC}} &= -\left(\mathcal{B}_{ij}^{kC}\right)^{-2}\frac{1}{n}(1-2\rho_{ij}^{kC}) \\ &\times\left[\begin{array}{l} b_k(x_i - z_j^C)^2 b_k b_k\left[\sum_{k\in\mathcal{S}} b_k \sum_{i\in\mathcal{N}_k} q_{ij}^{kC}\right]^{-2} \\ +e_k(t_i - a_j^C)^2 e_k e_k\left[\sum_{k\in\mathcal{S}} e_k \sum_{i\in\mathcal{N}_k} q_{ij}^{kC}\right]^{-2}\end{array}\right].\end{aligned} \tag{97}$$

Since the terms in the square bracket in (97) are all positive as is $\left(\mathcal{B}_{ij}^{kC}\right)^{-2}$, the sign of the slope of $(\mathcal{B}_{ij}^{kC})^{-1}$ in (97) depends only on whether $(1-2\rho_{ij}^{kC}) \lesseqgtr 0$. If $\rho_{ij}^{kC} < (>) 0.5$, so that $1-2\rho_{ij}^{kC} > (<) 0$, then the slope of $(\mathcal{B}_{ij}^{kC})^{-1}$ is *uniformly* negative (positive) over $\rho_{ij}^{kC} \in [0, 0.5)$ (over $\rho_{ij}^{kC} \in (0.5, 1]$) and zero at $\rho_{ij}^{kC} = 0.5$. Even though we *cannot* graph $(\mathcal{B}_{ij}^{kC})^{-1}$ precisely for voter i as a function of ρ_{ij}^{kC}, for the sake of argument and to illustrate $(\mathcal{B}_{ij}^{kC})^{-1}$ assume that it is a quadratic curve with a negative (positive) slope over $\rho_{ij}^{kC} \in [0, 0.5)$ (over $\rho_{ij}^{kC} \in (0.5, 1]$) and a zero slope at $\rho_{ij}^{kC} = 0.5$, as shown in Fig. 7.

Fig. 7 illustrates that the LHS and RHS of (96) cross twice. While for ρ_{ij}^{kC} to the left of the crossing closest to $\rho_{ij}^{kC} = 0$ the LHS is less than the RHS of (96), these values of ρ_{ij}^{kC} do not meet the necessary condition on voter i given in (63). Thus, ρ_{ij}^{kC} at the left-most intersection of LHS and RHS of (96) cannot be the sufficient pivotal vote share of voter i. The second crossing occurs at $\rho_{ij}^{kC} = \frac{1}{2}$. As Fig. 7 exemplifies the LHS is less than or equal to the RHS in (96) for $\rho_{ij}^{kC} \geqslant \frac{1}{2}$.

Define voter *i's sufficient state k pivotal probability for candidate j* at $(\mathbf{z}^C, \mathbf{a}^C)$, i.e.,

$$\psi_{ij}^{kC} = \frac{1}{2}. \tag{98}$$

So that the sufficient condition on voter i in state k for candidate j is that the probability that i votes for j be greater than or equal to i's sufficient pivotal probability, $\psi_{ij}^{kC} = \frac{1}{2}$ in (98),

$$\text{i.e., } \psi_{ij}^{kC} = \frac{1}{2} \leqslant \rho_{ij}^{kC}. \tag{99}$$

Since $\frac{Tr(\iota_k)}{Tr\left(\delta_{ij}^{kC}\right)} > 0$ for all $i \in \mathcal{N}_k$ and $k \in \mathcal{S}$, i's sufficient pivotal probability, ψ_{ij}^{kC} in (98) is greater than i's necessary pivotal probability for candidate j in state k, φ_{ij}^{kC} in (61), i.e., $\psi_{ij}^{kC} > \varphi_{ij}^{kC}$ for all $i \in \mathcal{N}_k$. For *some* voters $i \in \mathcal{N}_k$, the probability of voting for j may be *too low* in the sense that $\psi_{ij}^{kC} > \rho_{ij}^{kC}$, for these voters LHS is greater than the RHS in (96) or when LHS is less than the RHS in (96), the necessary condition given in (63) is not satisfied.

Having determined the sufficient condition on voter i for voters in state k, we now determine the sufficient condition under which j adopts (z_j^C, a_j^C) as its campaign in state k.

Aggregating the left hand side (LHS) of (99) over voters in state k and dividing by n_k gives *j's sufficient state k pivotal vote share* at $(\mathbf{z}^C, \mathbf{a}^C)$, i.e.,

$$\Psi_j^{kC} \equiv \Psi_j^k(\mathbf{z}^C, \mathbf{a}^C) = \frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\psi_{ij}^{kC} = \frac{1}{2}. \tag{100}$$

Note that Ψ_j^{kC} in (100) is constant across candidates and states and does not depend on j's electoral campaign, (z_j^C, a_j^C) or voters' preferences (x_i, t_i). Moreover, since $\frac{Tr(\iota_k)}{Tr\left(\delta_{ij}^{kC}\right)} > 0$ for all $i \in \mathcal{N}_k$ and $k \in \mathcal{S}$, j's sufficient state k pivotal vote share Ψ_j^{kC} is greater than Φ_j^{kC} in (64), i.e., $\Psi_j^{kC} > \Phi_j^{kC}$ for all $j \in \mathcal{C}$ and $k \in \mathcal{S}$.

Averaging the right hand side (RHS) of (99) over voters in state k, i.e., over $\mathcal{N}_k$, gives *j's expected state k vote share* at $(\mathbf{z}^C, \mathbf{a}^C)$, υ_j^{kC} in (6).

If (99) is satisfied for all $i \in \mathcal{N}_k$, i.e., if $\psi_{ij}^{kC} < \rho_{ij}^{kC}$ for all $i \in \mathcal{N}_k$, then $\Psi_j^{kC} < \upsilon_j^{kC}$. We now show that we do *not* require that all votes in state k vote for j with probability greater than $\frac{1}{2}$, that is, we do not need that (99) be satisfied for all $i \in \mathcal{N}_k$. To show this we partition the set of *voters in state k*, $\mathcal{N}_k$, into three subsets according to whether $\mathcal{A}_{ij}^{kC}$ in (94) is positive, negative or zero. Let $\mathcal{N}_k^{D+}$ be the set of voters in state k for whom $\mathcal{A}_{ij}^{kC} > 0$, i.e., for whom $\psi_{ij}^{kC} = \frac{1}{2} < \rho_{ij}^{kC}$; $\mathcal{N}_k^{D-}$ those for whom $\mathcal{A}_{ij}^{kC} < 0$, i.e., for whom $\psi_{ij}^{kC} = \frac{1}{2} > \rho_{ij}^{kC}$; and $\mathcal{N}_k^{D0}$ those for whom $\mathcal{A}_{ij}^{kC} = 0$, i.e., for whom $\psi_{ij}^{kC} = \frac{1}{2} = \rho_{ij}^{kC}$. So that when aggregating $\mathcal{A}_{ij}^{kC}$ over $\mathcal{N}_k^{D+}$, $\mathcal{N}_k^{D-}$ and $\mathcal{N}_k^{D0}$, we get that

$$\sum_{i\in\mathcal{N}_k^{D+}}\mathcal{A}_{ij}^{kC} > 0, \qquad \sum_{i\in\mathcal{N}_k^{D-}}\mathcal{A}_{ij}^{kC} > 0 \quad\text{and}\quad \sum_{i\in\mathcal{N}_k^{D0}}\mathcal{A}_{ij}^{kC} = 0.$$

Therefore, when summing $\mathcal{A}_{ij}^{kC}$ over all voters in state k, we have that

$$\begin{aligned}&\sum_{i\in\mathcal{N}_k}\left[1 - q_{ij}^{kC} f_{ij}^{kC} Tr\left((\Gamma_j^C)^{-1}\boldsymbol{\delta}_{ij}^{kC}\right)\right] \\ &\quad= \sum_{i\in\mathcal{N}_k}\mathcal{A}_{ij}^{kC} = \sum_{i\in\mathcal{N}_k^{D+}}\mathcal{A}_{ij}^{kC} + \sum_{i\in\mathcal{N}_k^{D-}}\mathcal{A}_{ij}^{kC},\end{aligned} \tag{101}$$

so that $\sum_{i\in\mathcal{N}_k}\mathcal{A}_{ij}^{kC} > 0$ iff $\left|\sum_{i\in\mathcal{N}_k^{D+}}\mathcal{A}_{ij}^{kC}\right| > \left|\sum_{i\in\mathcal{N}_k^{D-}}\mathcal{A}_{ij}^{kC}\right|$. When taking (101) into account in $\det(\mathbf{D}^2V_j^C)$ in (76), (101) says that if there are enough voters in state k for whom $\psi_{ij}^{kC} = \frac{1}{2} < \rho_{ij}^{kC}$ then, j's expected vote share in state k is high enough, i.e., higher than j's sufficient state k pivotal vote share, i.e., $\Psi_j^{kC} < \upsilon_j^{kC}$.

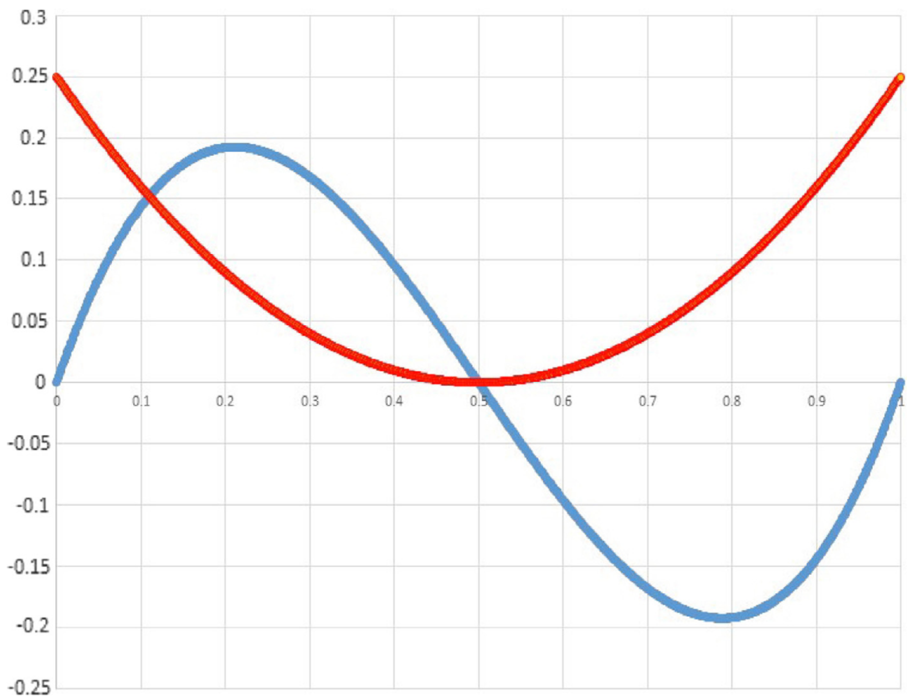

Fig. 7. Graph of the LHS (blue) and RHS (red) of (96). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

The *sufficient state k condition* for j to adopt (z_j^C, a_j^C) as its campaign in state k is that

$$\Psi_j^{kC} = \frac{1}{2} < v_j^{kC}. \tag{102}$$

Having determined the conditions under which j adopts its campaign in state k, we now examine the conditions under which $\det\left(\mathbf{D}^2 V_j^C\right) = \sum_{k\in\mathcal{S}} \frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\mathcal{A}_{ij}^{kC}$ in (76) is positive.

Multiply the LHS of (102) by state k's share of the national population, $\frac{n_k}{n}$, and sum over all states to obtain j's *sufficient national pivotal vote share*, η_j^C at $(\mathbf{z}^C, \mathbf{a}^C)$, i.e.,

$$\eta_j^C \equiv \eta_j(\mathbf{z}^C, \mathbf{a}^C) \equiv \sum_{k\in\mathcal{S}} \frac{n_k}{n}\Psi_{ij}^{kC} = \sum_{k\in\mathcal{S}} \frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\psi_{ij}^{kC}$$
$$= \frac{1}{2}. \tag{103}$$

Similarly, multiplying the RHS of (102) by state k's share of the national population, $\frac{n_k}{n}$, and summing over all states gives j's *national expected vote share*, V_j^C in (7) or in (5).

If $\sum_{i\in\mathcal{N}_k}\mathcal{A}_{ij}^{kC}$ in (76) is positive for all $k \in \mathcal{S}$ where $\mathcal{A}_{ij}^{kC}$ is given by (94) then from (102) we know that $\Psi_j^{kC} < v_j^{kC}$ for all $k \in \mathcal{S}$, so that j's national vote share at $(\mathbf{z}^C, \mathbf{a}^C)$, V_j^C in (7), is greater than j's national pivotal vote share, η_j^C in (66), i.e., $\eta_j^C = \frac{1}{2} < V_j^C$, implying that $\det(\mathbf{D}^2 V_j^C) = \sum_{k\in\mathcal{S}}\frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\mathcal{A}_{ij}^{kC} > 0$. For some states, $\sum_{i\in\mathcal{N}_k}\mathcal{A}_{ij}^{kC}$ may be negative, so that in these states the probability that state k votes for j is *too low*, i.e., $\Psi_j^{kC} > v_j^{kC}$.

Since $\sum_{i\in\mathcal{N}_k}\mathcal{A}_{ij}^{kC}$ may be positive or negative, we partition the set of *states* $\mathcal{S}$ into three subsets. Let $\mathcal{S}^{D+}$ be the set of states for whom $\sum_{i\in\mathcal{N}_k}\mathcal{A}_{ij}^{kC} > 0$, i.e., for whom $\Psi_j^{kC} < v_j^{kC}$; $\mathcal{S}^{D-}$ those for whom $\sum_{i\in\mathcal{N}_k}\mathcal{A}_{ij}^{kC} < 0$, i.e., those for whom $\Psi_j^{kC} > v_{ij}^{kC}$; and $\mathcal{S}^{D0}$ those for whom $\sum_{i\in\mathcal{N}_k}\mathcal{A}_{ij}^{kC} = 0$ i.e., for whom $\Psi_j^{kC} = v_j^{kC}$.

Given that state k's share of the national population $\frac{n_k}{n}$ is positive for all $k \in \mathcal{S}$, the sign of $\frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\mathcal{A}_{ij}^{kC}$ depends only on the sign of $\sum_{i\in\mathcal{N}_k}\mathcal{A}_{ij}^{kC}$. Thus, when aggregating $\frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\mathcal{A}_{ij}^{kC}$ over $\mathcal{S}^{D+}$, $\mathcal{S}^{D-}$ and $\mathcal{S}^{D0}$, we get that

$$\sum_{k\in\mathcal{S}^{D+}}\frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\mathcal{A}_{ij}^{kC} > 0, \qquad \sum_{k\in\mathcal{S}^{D-}}\frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\mathcal{A}_{ij}^{kC} < 0$$

and $\sum_{k\in\mathcal{S}^{D0}}\frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\mathcal{A}_{ij}^{kC} = 0.$

The determinant of $\mathbf{D}^2 V_j^C$ in (76) can be re-written as

$$\det(\mathbf{D}^2 V_j^C)$$
$$= \det\left[\mathbf{R}_j^C\right] + 4\det(\Gamma_j^C)$$
$$\times\left\{\sum_{k\in\mathcal{S}^{D+}}\frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\mathcal{A}_{ij}^{kC} + \sum_{k\in\mathcal{S}^{D-}}\frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\mathcal{A}_{ij}^{kC}\right\}.$$

Since $\det\left[\mathbf{R}_j^C\right] = 0$ and we know from (72) that $\det(\Gamma_j^C) > 0$ we have that

$$\det(\mathbf{D}^2 V_j^C)$$
$$= 4\det(\Gamma_j^C)\left\{\sum_{k\in\mathcal{S}^{D+}}\frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\mathcal{A}_{ij}^{kC} + \sum_{k\in\mathcal{S}^{D-}}\frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\mathcal{A}_{ij}^{kC}\right\}.$$

Thus, $\det(\mathbf{D}^2 V_j^C) > 0$ iff $\left|\sum_{k\in\mathcal{S}^{D+}}\frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\mathcal{A}_{ij}^{kC}\right| > \left|\sum_{k\in\mathcal{S}^{D-}}\frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k}\mathcal{A}_{ij}^{kC}\right|$. Therefore, if $\sum_{i\in\mathcal{N}_k}\mathcal{A}_{ij}^{kC} > 0$ for *enough* states, i.e., if $\Psi_j^{kC} < v_j^{kC}$ in (102) is satisfied for *enough states* then $\det\left(\mathbf{D}^2 V_j^C\right) > 0$. This says that if there are enough states from whom the probability of voting for j is high enough, i.e., if j's state k vote share v_j^{kC} is higher than j's state k pivotal vote share, Ψ_j^{kC} in (100) for enough states then $\eta_j^C = \frac{1}{2} < V_j^C$, and $\det\left(\mathbf{D}^2 V_j^C\right) > 0$.

Thus, if $\Psi_j^{kC} < v_j^{kC}$ in (65) is satisfied for *enough states* then $\det\left(\mathbf{D}^2 V_j^C\right) > 0$. So that, after averaging Ψ_j^{kC} and v_j^{kC} over all states, we obtain that the *sufficient national condition* for j to adopt (z_j^C, a_j^C)

as its electoral campaign – when all other candidates adopt their critical campaigns, i.e., holding $(\mathbf{z}^C_{-j}, \mathbf{a}^C_{-j})$ constant – is that

$$\eta_j^C = \frac{1}{2} \leqslant V_j^C. \tag{104}$$

This condition says that when j's national vote share at $(\mathbf{z}^C, \mathbf{a}^C)$, V_j^C, is *higher* than or equal to j's sufficient pivotal vote share at $(\mathbf{z}^C, \mathbf{a}^C)$, $\eta_j^C = \frac{1}{2}$, then $\det\left(\mathbf{D}^2 V_j^C\right) > 0$.

Note that j's sufficient national pivotal vote share, $\eta_j^C = \frac{1}{2}$ in (103), is always greater than j's necessary national pivotal vote share, $\vartheta_j^C \equiv \frac{1}{2} - \frac{1}{4}\sum_{k\in\mathcal{S}} \frac{n_k}{n}\frac{1}{n_k}\sum_{i\in\mathcal{N}_k} \frac{Tr(\iota_k)}{Tr\left(\delta_{ij}^{kC}\right)}$ in (66), i.e., that $\eta_j^C > \vartheta_j^C$ for all $j \in \mathcal{C}$.

Recall that $\sum_{i\in\mathcal{N}_k} \mathcal{A}_{ij}^{kC} > 0$ only if (99) is satisfied for enough voters in these states. Thus, the probability that a state votes for candidate j must be high enough in enough states and this happens only if the probability that voters in these states vote for j is high enough for enough voters in these states. ■

For the necessary part of the proof, assume $(\mathbf{z}^C, \mathbf{a}^C)$ is a *weak* local Nash equilibrium of the election. Then for all $j \in \mathcal{C}$, j's Hessian matrix evaluated at (z_j^C, a_j^C) must be negative semi-definite implying that $Tr(\mathbf{D}^2 V_j^C) \leq 0$ which only happens when for enough states $Tr(\mathbf{G}_j^{kC}) < 0$. Moreover, $Tr(\mathbf{G}_j^{kC}) < 0$ in state k only iff $Tr(c_j^{kC}) \leq 0$ for enough voters in these states. If for some $j \in \mathcal{C}$, $Tr(\mathbf{D}^2 V_j^C) > 0$, then it must be the case that there are enough states for whom $Tr(\mathbf{G}_j^{kC}) > 0$ so that in these states there are enough voters for whom $Tr(c_{ij}^{kC}) > 0$ which violates the weak Nash equilibrium condition.

We can restate the necessary part of the proof in terms of expected and pivotal vote shares and probabilities. Assume that $(\mathbf{z}^C, \mathbf{a}^C)$ is a *weak* local Nash equilibrium of the election. Then $Tr(\mathbf{D}^2 V_j^C) \leq 0$ when j's expected national vote share, V_j^C in (4) or in (7), is greater than j's necessary national pivotal vote share, ϑ_j^C in (66), i.e., iff $V_j^C > \vartheta_j^C$. For $V_j^C > \vartheta_j^C$ there must be enough states for whom the expected vote share in that state, v_j^{kC} in (6), is greater than j's necessary state k pivotal vote share, Φ_j^C in (64), i.e., $v_j^{kC} > \Phi_j^C$. Now, in state k, for $v_j^{kC} > \Phi_j^C$ it must be the case that there are enough voters in state k for whom the probability of voting for j, ρ_{ij}^{kC} in (16), is greater than i's pivotal probability φ_{ij}^C in (61), i.e., for whom $\rho_{ij}^{kC} > \varphi_{ij}^C$. If $\vartheta_j^C < V_j^C$ for some $j \in \mathcal{C}$ then it must be the case that there are enough states for whom voting for j is low enough, i.e., for whom $v_j^{kC} < \Phi_j^C$, and this happens only when in these states there are enough voters for whom the probability of voting for j is low enough, i.e., for whom $\rho_{ij}^{kC} < \varphi_{ij}^C$ which violates the weak Nash equilibrium condition. This completes the proof of necessity. ■

References

Ansolabehere, S., de Figueiredo, J.M., Snyder, J.M., 2003. Why is there so little money in U.S. politics? J. Econom. Perspect. 17 (1), 105–130.

Ansolabehere, S., Snyder, J.M., 2000. Valence politics and equilibrium in spatial election models. Public Choice 103, 327–336.

Ashworth, S., Bueno de Mesquita, E., 2009. Elections with platform and valence competition. Games Econom. Behav. 67, 191–216.

Aragones, E., Palfrey, T., 2002. Mixed equilibrium in a Downsian model with a favored candidate. J. Econom. Theory 103, 131–161.

Austen-Smith, D., Banks, J.S., 1999. Cycling of simple rules in the spatial model. Soc. Choice Welf. 16 (4), 663–672.

Brams, S.J., Davis, M.D., 1973. Resource-allocation models in Presidential campaigning: Implications for democratic representation. Ann. New York Acad. Sci. 219 (1), 105–123.

Clarke, H.D., Kornberg, A., Scotto, T.J., 2009. Making Political Choices. Toronto University Press, Toronto.

Condorcet, J., 1984 [1785]. Essai sur l'application de l'analyse a la probabilite des decisions rendus a la pluralite des voix. Imprimerie Royale, Paris.

Downs, A., 1957. An Economic Theory of Democracy. Harper and Row, New York.

Gallego, M., Schofield, N., 2013. The convergence coefficient across political systems. Sci. World J. 2013, Article ID 653035.

Gallego, M., Schofield, N., 2016a. Do parties converge to the electoral mean in all political systems? J. Theoret. Polit. 28 (2), 288–330.

Gallego, M., Schofield, N., 2016b. Modelling the effect of campaign advertising on US presidential elections. In: Gallego, M., Schofield, N. (Eds.), The Political Economy of Social Choices. Springer, pp. 123–153.

Gallego, M., Schofield, N., McAlister, K., Jeon, J.S., 2014. The variable choice set logit model applied to the 2004 Canadian election. Public Choice 158 (3–4), 427–463.

Groseclose, T., 2001. A model of candidate location when one candidate has a valence advantage. Amer. J. Polit. Sci. 45 (4), 862–886.

Hendricks, J.A., Kaid, L.L., 2011. Techno Politics in Presidential Campaigning: New Voices, New Technologies, and New Voters. Routledge, New York.

Herrerra, H., Levine, D.K., Martinelli, C., 2008. Policy platforms, campaign spending and voter participation. J. Publ. Econ. 92, 501–513.

Kim, J.H., Schofield, N., 2016. Spatial model of US presidential election in 2012. In: Schofield, N., Gallego, M. (Eds.), The Political Economy of Social Choices. Springer, Heidelberg, pp. 233–242.

Labzina, E., Schofield, N., 2014. Application of the variable choice logit model to the British general election of 2010. In: Schofield, N., Caballero, G. (Eds.), The Political Economy of Governance: Institutions, Political Performance and Elections. Springer, Heidelberg, pp. 313–334.

Maddison, J., 1999 [1787]. The Federalist No. 10. In: Rakove, J. (Ed.), James Madison: Writings. The Library of America, New York.

McKelvey, R.D., 1976. Intransitivities in multidimensional voting models and some implications for agenda control. J. Econom. Theory 12 (3), 472–482.

McKelvey, R.D., Patty, J., 2006. A theory of voting in large elections. Games Econom. Behav. 57, 155–180.

Meirowitz, A., 2008. Electoral contests, incumbency advantages and campaign finance. J. Polit. 70, 681–699.

Riker, W.H., 1964. Federalism: Origin, Operation, Maintenance. Little Brown, Boston. MA.

Riker, W.H., 1980. Implications from the disequilibrium of majority rule for the study of institutions. Amer. Polit. Sci. Rev. 74, 432–446.

Riker, W.H., 1982. The two party system and Duverger's Law: An essay on the history of political science. Amer. Polit. Sci. Rev. 76, 753–766.

Riker, W.H., 1986. The Art of Political Manipulation. Yale University Press, New Haven.

Riker, W.H., 1987. The Development of American Federalism. Kluwer, Boston, MA.

Saari, D., 1997. The Generic existence of a core for q-Rules. Econom. Theory 9, 219–260.

Sanders, D., Clarke, H., Stewart, M.C., Whiteley, P., 2011. Downs, Stokes and the dynamics of electoral choice. British J. Polit. Sci. 41 (2), 287–314.

Schofield, N., 1978. Instability of simple dynamic games. Rev. Econom. Stud. 45, 575–594.

Schofield, N., 2002. Evolution of the constitution. British J. Polit. Sci. 32, 1–20.

Schofield, N., 2006. Equilibria in the spatial stochastic model of voting with party activists. Rev. Econ. Des. 10 (3), 183–203.

Schofield, N., 2007. The mean voter theorem: Necessary and sufficient conditions for convergent equilibrium. Rev. Econom. Stud. 74, 965–980.

Schofield, N., Gallego, M., Jeon, J.S., 2011a. Leaders, voters and activists in elections in the Great Britain 2005 and 2010. Electoral Stud. 30 (3), 484–496.

Schofield, N., Claassen, C., Gallego, M., Ozdemir, U., 2011b. Empirical and formal models of the US presidential elections in 2004 and 2008. In: Schofield, N., Caballero, G. (Eds.), Political Economy of Institutions, Democracy and Voting. Springer, Berlin, pp. 217–258.

Snyder, J.M., 1989. Election goals and the allocation of campaign resources. Econometrica 57 (3), 637–660.

Stokes, D., 1963. Spatial models and party competition. Amer. Polit. Sci. Rev. 57, 368–377.

Stokes, D., 1992. Valence politics. In: Kavanagh, D. (Ed.), Electoral Politics. Clarendon Press, Oxford, pp. 141–164.

Strömberg, D., 2008. How the Electoral College influences campaigns and policy: the probability of being Florida. Amer. Econom. Rev. 98 (3), 769–807.

Train, K., 2003. Discrete Choice Methods for Simulation. Cambridge University Press, Cambridge, UK.

Zakharov, A.V., 2009. A model of candidate location with endogenous valence. Public Choice 138, 347–366.