1 Technical Appendix

1.1 The Spatial Valence Model

Instead of using a model based on committee bargaining, as in the first section of this chapter we can instead adapt Hinich's 1977) earlier stochastic model of voting.

Definition 1. The Stochastic Vote Model $\mathbb{E}(\boldsymbol{\lambda},\boldsymbol{\mu},\beta;\Psi)$ with Activist Valence. Voters are characterized by a distribution, $\{x_i \in W : i \in V\}$, of voter ideal points for the members of the electorate, V, of size v. We assume that W is an open, convex subset of Euclidean space, $\mathbb{R}^w$, with w finite. Each of the parties in the set $N = \{1,\ldots,j,\ldots,n\}$ chooses a policy, $z_j \in W$, to declare. Let $\mathbf{z} = (z_1,\ldots,z_n) \in W^n$ be a typical vector of party policy positions.

Given $\mathbf{z}$, each voter, i, is described by a vector

$$\begin{aligned}\mathbf{u}_i(x_i,\mathbf{z}) &= (u_{i1}(x_i,z_1),\ldots,u_{ip}(x_i,z_n)), \text{ where} \\ u_{ij}(x_i,z_j) &= \lambda_j + \mu_j(z_j) - \beta||x_i - z_j||^2 + \epsilon_j \\ &= u^*_{ij}(x_i,z_j) + \epsilon_j.\end{aligned}$$

Here $u^*_{ij}(x_i,z_j)$ is the observable component of utility. The term, λ_j, is the fixed or *exogenous valence* of party j, while the function $\mu_j(z_j)$ is the component of valence generated by activist contributions to party j. The term β is a positive constant, called the *spatial parameter*, giving the importance of policy difference defined in terms of the Euclidean norm, $||\cdot||$, on W. The error vector $\epsilon = (\epsilon_1,\ldots,\epsilon_j,\ldots,\epsilon_n)$ is generated by the *Type I Extreme value Distribution*, also known as the *multinomial logit*.

Voter behavior is modeled by a probability vector. The probability that a voter i chooses party j at the vector $\mathbf{z}$ is

$$\begin{aligned}\rho_{ij}(\mathbf{z}) &= \Pr[[u_{ij}(x_i,z_j) > u_{il}(x_i,z_l)], \text{ for all } l \neq j]. &(1)\\ &= \Pr[\epsilon_l - \epsilon_j < u^*_{ij}(x_i,z_j) - u^*_{il}(x_i,z_j), \text{ for all } l \neq j]. &(2)\end{aligned}$$

Here Pr stands for the probability operator generated by the distribution assumption on ϵ.

The *expected vote-share* of agent j generated by the model $\mathbb{E}(\boldsymbol{\lambda},\boldsymbol{\mu};\beta;\Psi)$ is

$$E_j(\mathbf{z}) = \frac{1}{v}\sum_{i\in V}\rho_{ij}(\mathbf{z}). \quad (3)$$

Definition.2. The Type I Extreme Value Distribution, Ψ.

The cumulative distribution,Ψ, has the closed form

$$\Psi(x) = \exp\left[-\exp\left[-x\right]\right],$$

with probability density function

$$\psi(x) = \exp[-x]\exp\left[-\exp\left[-x\right]\right]$$

With this distribution assumption, it follows, for each voter i, and party j, that

$$\rho_{ij}(\mathbf{z}) = \frac{\exp[u^*_{ij}(x_i, z_j)]}{\sum_{k=1}^{n} \exp u^*_{ik}(x_i, z_k)}. \tag{4}$$

Definition.3. Equilibrium Concepts.

(i) A strategy vector $\mathbf{z}^*=(z^*_1, \ldots, z^*_{j-1}, z^*_j, z^*_{j+1}, \ldots, z^*_n) \in W^n$ is a *local strict Nash equilibrium* (LSNE) for the function $E : W^n \to \mathbb{R}^n$ iff, for each party $j \in N$, there exists a neighborhood W_j of z^*_j in W such that

$$\begin{array}{rcl} E_j(z^*_1, \ldots, z^*_{j-1}, z^*_j, z^*_{j+1}, \ldots, z^*_p) & > & E_j(z^*_1, \ldots, z_j, \ldots, z^*_n) \\ \text{for all } z_j & \in & W_j - \{z^*_j\}. \end{array}$$

(ii) A strategy vector $\mathbf{z}^*=(z^*_1, \ldots, z^*_{j-1}, z^*_j, z^*_{j+1}, \ldots, z^*_n)$ is a *local weak Nash equilibrium* (LNE) iff, for each agent j,there exists a neighborhood W_j of z^*_j in W such that

$$\begin{array}{rcl} E_j(z^*_1, \ldots, z^*_{j-1}, z^*_j, z^*_{j+1}, \ldots, z^*_n) & \geq & E_j(z^*_1, \ldots, z_j, \ldots, z^*_n) \\ \text{for all } z_j & \in & W_j. \end{array}$$

(iii) A strategy vector $\mathbf{z}^*=(z^*_1, \ldots, z^*_{j-1}, z^*_j, z^*_{j+1}, \ldots, z^*_n)$ is a *strict* or *weak*, *pure strategy Nash equilibrium* (PSNE or PNE) iff W_j can be replaced by W in (i) and (ii) respectively.

Obviously if $\mathbf{z}^*$ is an LSNE or a PNE it must be an LNE, while if it is a PSNE then it must be an LSNE. We use the notion of LSNE to avoid problems with the degenerate situation when there is a zero eigenvalue to the Hessian. The weaker requirement of LNE allows us to obtain a necessary condition for $\mathbf{z}^*$ to be an LNE and thus a PNE, without having to invoke concavity. Of particular interest is the joint mean vector

$$x^* = \frac{1}{v} \sum_{i \in V} x_i. \tag{5}$$

We first transform coordinates so that in the new coordinate system, $x^* = 0$. We shall refer to $\mathbf{z}_0 = (0, \ldots, 0)$ as the *joint origin.*

Theorem 1 below shows that $\mathbf{z}_0 = (0, \ldots, 0)$ will generally not satisfy the first-order condition for an LSNE, namely that the differential of E_j, with respect to z_j be zero. However, if the activist valence function is identically zero, so that only exogenous valence is relevant, then the first-order condition will be satisfied. On the other hand, Theorem 2 shows that there are necessary and sufficient conditions for $\mathbf{z}_0$ to be an LSNE in a model without activist valence.

A corollary of Theorem .2 gives these conditions in terms of a "convergence coefficient" determined by the Hessian of party 1, with the lowest valence.

It follows from the model that for voter i, with bliss point, x_i, the probability, $\rho_{ij}(\mathbf{z})$, that i picks j at $\mathbf{z}$ is given by

$$\rho_{ij}(\mathbf{z}) = [1 + \Sigma_{k\neq j}[\exp(f_{jk})]]^{-1}, \tag{6}$$

where $f_{jk} = \lambda_k + \mu_k(z_k) - \lambda_j - \mu_j(z_j) + \beta||x_i - z_j||^2 - \beta||x_i - z_k||^2.$

Theorem 1 below shows that the first-order condition for $\mathbf{z}^*$ to be an LSNE is that it be a *balance solution.*

Definition .4: The Balance Solution for the Model $\mathbb{E}(\boldsymbol{\lambda}, \boldsymbol{\mu};\beta;\Psi)$

(i) Let $[\rho_{ij}(\mathbf{z})] = [\rho_{ij}]$ be the matrix of voter probabilities at the vector $\mathbf{z}$, and let

$$[\alpha_{ij}] = \frac{\rho_{ij} - \rho_{ij}^2}{\Sigma_k^v(\rho_{kj} - \rho_{kj}^2)}$$

be the matrix of coefficients. The *balance equation* for z_j^* is given by expression

$$z_j^* = \frac{1}{2\beta}\frac{d\mu_j}{dz_j} + \sum_{i=1}^{v}\alpha_{ij}x_i. \tag{7}$$

(ii) The vector $\sum_i \alpha_{ij}x_i$ is called the *weighted electoral mean* for party j, and can be written

$$\sum_{i=1}^{v}\alpha_{ij}x_i = \frac{d\mathcal{E}_j^*}{dz_j}. \tag{8}$$

(iii) The balance equation for party j can then be rewritten as

$$\left[\frac{d\mathcal{E}_j^*}{dz_j} - z_j^*\right] + \frac{1}{2\beta}\frac{d\mu_j}{dz_j} = 0. \tag{9}$$

(iv) The bracketed term on the left of this expression is termed the *marginal electoral pull of party* j and is a gradient vector pointing towards the weighted electoral mean. This weighted electoral mean is that point where the electoral pull is zero. The vector $\frac{d\mu_j}{dz_j}$ is called *the marginal activist pull for party* j.

(v) If the vector $\mathbf{z}^* = (z_1^*, \ldots, z_j, \ldots, z_n^*)$ satisfies the set of balance equations, $j = 1, \ldots, n$, then call $\mathbf{z}^*$ the *balance solution.*

Theorem 1 is proved in Schofield (2006).

Theorem .1. Consider the electoral model $\mathbb{E}(\boldsymbol{\lambda}, \boldsymbol{\mu};\beta;\Psi)$ based on the Type I extreme value distribution, and including both exogenous and activist valences. The first-order condition for z* to be an LSNE is that it is a balance solution. If all activist valence functions are highly concave, in the sense of having negative eigenvalues of sufficiently great magnitude, then the balance solution will be a PNE.

In the case that all activist valence functions $\{\mu_j\}$ are identically zero, we write the electoral model as $\mathbb{E}(\boldsymbol{\lambda};\beta;\Psi)$. Then by Theorem .1, the coefficients, α_{ij}, are independent of i. Thus, when there is only exogenous valence, the balance condition gives

$$z_j^* = \frac{1}{v}\sum_{i=1}^{v} x_i. \tag{10}$$

By a change of coordinates we can choose $\Sigma x_i = 0$. In this case, the marginal electoral pull is zero at the origin and the joint origin $\mathbf{z}_0 = (0, \ldots, 0)$ satisfies the first-order condition.

To characterize the variation in voter preferences, we represent in a simple form the covariance matrix, ∇_0, given by the distribution of voter ideal points.

Definition .5. The Electoral Covariance Matrix, ∇_0.

Let $W = \mathbb{R}^w$ be endowed with a system of coordinate axes $r = 1, \ldots, w$. For each coordinate axis let $\xi_r = (x_{1r}, x_{2r}, \ldots, x_{vr})$ be the vector of the r^{th} coordinates of the set of v voter bliss points. The scalar product of ξ_r and ξ_s is denoted by (ξ_r, ξ_s).

(i) The symmetric $w \times w$ *electoral covariance matrix* about the origin is denoted ∇_0 and is defined by

$$\nabla_0 = \frac{1}{v}\left[(\xi_r, \xi_s)\right]_{s=1,\ldots,w}^{r=1,\ldots,w}.$$

(ii) Let $(\sigma_r, \sigma_s) = \frac{1}{v}(\xi_r, \xi_s)$ be the electoral covariance between the r^{th} and s^{th} axes, and $\sigma_s^2 = \frac{1}{v}(\xi_s, \xi_s)$ be the electoral variance on the s^{th} axis, with

$$\sigma^2 = \sum_{s=1}^{w} \sigma_s^2 = \frac{1}{v}\sum_{s=1}^{w}(\xi_s, \xi_s) = trace(\nabla_0)$$

the total electoral variance.

At the vector $\mathbf{z}_0 = (0, \ldots, 0)$ the probability $\rho_{ij}(\mathbf{z}_0)$ that i votes for party j is independent of i, and is given by

$$\rho_j = \left[1 + \sum_{k \neq j} \exp\left[\lambda_k - \lambda_j\right]\right]^{-1}. \tag{11}$$

Definition.6: The Convergence Coefficient of the Model $\mathbb{E}(\boldsymbol{\lambda};\beta;\Psi)$.

The *characteristic matrix* for party j is

$$C_j = \left[2A_j\nabla_0 - I\right], \tag{12}$$

where I is the w by w identity matrix.

(iii) The *convergence coefficient of the model* is

$$c(\boldsymbol{\lambda};\beta;\Psi) = 2\beta[1-2\rho_1]\sigma^2 = 2A_1\sigma^2. \tag{13}$$

At the vector $\mathbf{z}_0 = (0,\ldots,0)$ the probability $\rho_{ij}(\mathbf{z}_0)$ that i votes for party j is independent of i, and is given by (11). Thus, if all valences are identical then $\rho_j = \frac{1}{n}$, for all j, as expected. The effect of increasing λ_j, for $j \neq 1$, is clearly to decrease ρ_1, and therefore to increase A_1, and thus $c(\boldsymbol{\lambda},\beta;\Psi)$. Schofield (2007) proves the following theorem.

Theorem 2.. The necessary condition for the joint origin to be an LSNE in the model $\mathbb{E}(\boldsymbol{\lambda};\beta;\Psi)$ is that the characteristic matrix

$$C_1 = [2A_1\nabla_0 - I]$$

of the party 1, with lowest valence, has negative eigenvalues. ▮

Theorem 2. immediately gives the following corollaries:

Corollary 3. Consider the model $\mathbb{E}(\boldsymbol{\lambda};\beta;\Psi)$. In the case that X is w-dimensional, then the necessary condition for the joint origin to be an LNE is that $c(\boldsymbol{\lambda};\beta;\Psi) \leq w$.
Ceteris paribus, an LNE at the joint origin is "less likely" the greater are the parameters β, $\lambda_p - \lambda_1$ and σ^2.

Corollary 4. In the two-dimensional case, a sufficient condition for the joint origin to be an LSNE for the model $\mathbb{E}(\boldsymbol{\lambda};\beta;\Psi)$ is that $c(\lambda;\beta;\Psi) < 1$.

It is evident that sufficient conditions for existence of an LSNE at the joint origin in higher dimensions can be obtained using standard results on the determinants, $\{\det(C_j)\}$, and traces, $\{\text{trace}(C_j)\}$, of the characteristic matrices.

Notice that the case with two parties of equal valence immediately gives a situation with $2\beta[1-2\rho_1]\sigma^2 = 0$, irrespective of the other parameters. However, if $\lambda_2 > \lambda_1$, then the joint origin may fail to be an LNE if $\beta\sigma^2$ is sufficiently large.